UNIMODALITY AND YOUNG’S LATTICE

Dennis Stanton†

Abstract. Young’s lattice of a partition \( \lambda \) consists of all partitions whose Ferrers diagrams fit inside \( \lambda \). Several infinite families of partitions are given whose Young’s lattice is not rank unimodal. Some related problems are discussed.

1. Introduction.
   It is well known that the \( q \)-binomial coefficient
   \[
   \binom{n + m}{m}_q
   \]
   is a symmetric unimodal polynomial in \( q \) (see, e.g. [1, §3.5]). Recall that a sequence of integers \( a_i \) is unimodal if there exists an integer \( N \) such that
   \[
   a_0 \leq a_1 \leq \cdots \leq a_N \geq a_{N+1} \geq a_{N+2} \geq \cdots .
   \]
   A polynomial is called unimodal if its sequence of coefficients is unimodal. For the \( q \)-binomial coefficient in (1.1), \( N = nm/2 \), half of the degree of the polynomial.

   Combinatorially, the \( q \)-binomial coefficient has the following interpretation. If \( a_i \) is the number of partitions of \( i \) which lie inside an \( n \times m \) rectangle, then \( a_i \) is the coefficient of \( q^i \) in (1.1). This is another way of saying that the \( q \)-binomial coefficient is the generating function for all partitions which lie inside an \( n \times m \) rectangle. These partitions are the elements of a lattice called Young’s lattice, whose order relation is given by containment of the respective Ferrers diagrams.

   Instead of a rectangle, we can consider Young’s lattice for any partition \( \lambda \). Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1 \), and call the lattice \( Y_\lambda \). The purpose of this paper is to study the unimodality properties of \( Y_\lambda \).

   We let \( G(Y_\lambda)(q) \) be the generating function for all partitions which lie inside \( \lambda \). If \( \lambda' \) denotes the conjugate of the partition \( \lambda \), it is clear that
   \[
   G(Y_\lambda)(q) = G(Y_{\lambda'})(q).
   \]
   We will call a partition \( \lambda \) unimodal if \( G(Y_\lambda)(q) \) is a unimodal polynomial. Note that the non-unimodality of \( \lambda \) is equivalent to the following condition on the coefficients \( b_i \) of \( (1-q)G(Y_\lambda)(q) \). There is some \( i < j \) satisfying \( b_i < 0 \) and \( b_j > 0 \).

   In §2 we give the data from the programs which were written for this problem. The theorems which are suggested from the data are stated and proved in §3. Some final remarks, including observations and conjectures, are given in §4. We will use the notation \( \lfloor x \rfloor \) and \( \lceil x \rceil \) for the greatest integer \( \leq x \) and the least integer \( \geq x \) respectively.

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2. Data.

All partitions of $n \leq 50$ were tested. (There are $204226$ partitions of $50$.) Not all partitions $\lambda$ are unimodal. The first non-unimodal $\lambda$ is $\lambda = (8, 8, 4, 4)$, with coefficients

$$1 1 2 3 5 6 9 11 15 17 21 23 27 28 31 30 31 27 24 18 14 8 5 2 1.$$

It is true that all partitions of $n \leq 23$, or all partitions which lie inside a $7 \times 7$ square, are unimodal. The following table lists the non-unimodal partitions of $n \leq 36$. Because of (1.2), we list only one of $\lambda$ and $\lambda'$. The value of $i$ for which unimodality fails, and the three offending values $a_{i-1}, a_i$, and $a_{i+1}$ are also given.

<table>
<thead>
<tr>
<th>Partition</th>
<th>i</th>
<th>Values</th>
<th>Partition</th>
<th>i</th>
<th>Values</th>
</tr>
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<tr>
<td>8 8 4 4</td>
<td>15</td>
<td>31 30 31</td>
<td>11 11 6 6</td>
<td>21</td>
<td>67 66 67</td>
</tr>
<tr>
<td>10 9 4 4</td>
<td>17</td>
<td>46 45 46</td>
<td>14 13 4 4</td>
<td>21</td>
<td>76 75 76</td>
</tr>
<tr>
<td>10 10 4 4</td>
<td>17</td>
<td>46 45 46</td>
<td>16 12 4 4</td>
<td>23</td>
<td>91 90 91</td>
</tr>
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<td>12 10 4 4</td>
<td>19</td>
<td>61 60 61</td>
<td>14 14 4 4</td>
<td>21</td>
<td>76 75 76</td>
</tr>
<tr>
<td>12 11 4 4</td>
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<td>61 60 61</td>
<td>12 12 4 4</td>
<td>23</td>
<td>81 80 81</td>
</tr>
<tr>
<td>12 12 4 4</td>
<td>19</td>
<td>61 60 61</td>
<td>12 10 8 6</td>
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<td>21</td>
<td>76 75 76</td>
<td></td>
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</tr>
</tbody>
</table>

Table 1

Many of the partitions on the previous list have the form $\lambda = (a, a, b, b)$. The following table lists all non-unimodal partitions of this form with $a \leq 24$.

<table>
<thead>
<tr>
<th>Partition</th>
<th>i</th>
<th>Values</th>
<th>Partition</th>
<th>i</th>
<th>Values</th>
</tr>
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<td>8 8 4 4</td>
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<td>31 30 31</td>
<td>20 20 4 4</td>
<td>27</td>
<td>121 120 121</td>
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<td>10 10 4 4</td>
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<td>46 45 46</td>
<td>20 20 10 10</td>
<td>37</td>
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<td>11 11 6 6</td>
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<td>67 66 67</td>
<td>20 20 12 12</td>
<td>39</td>
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<td>123 122 123</td>
<td>22 22 4 4</td>
<td>29</td>
<td>136 135 136</td>
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<td>16 16 4 4</td>
<td>23</td>
<td>91 90 91</td>
<td>22 22 11 11</td>
<td>41</td>
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</tr>
<tr>
<td>16 16 9 9</td>
<td>31</td>
<td>173 172 173</td>
<td>22 22 13 13</td>
<td>43</td>
<td>405 404 406</td>
</tr>
<tr>
<td>17 17 8 8</td>
<td>31</td>
<td>188 187 188</td>
<td>23 23 8 8</td>
<td>37</td>
<td>323 322 323</td>
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<td>33</td>
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<td>18 18 10 10</td>
<td>35</td>
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<tr>
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<td>273 272 273</td>
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Table 2

It is also of interest to test $\lambda = (a, a, b, b)$ for particular values of $a$. Table 3 takes $a = 90$ and $a = 89$.

<table>
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<tr>
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<th>Values</th>
<th>Partition</th>
<th>i</th>
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<td>21123 21117 21123</td>
</tr>
</tbody>
</table>

Table 3
Table 3

Table 4 gives the number of partitions of $n$ ($p(n)$) and the number of non-unimodal partitions of $n$ ($NU(n)$) for $n \leq 50$.

<table>
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<tr>
<th>n</th>
<th>p(n)</th>
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<th>p(n)</th>
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<td>14883</td>
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<td>75175</td>
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<td>40</td>
<td>34</td>
<td>123</td>
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</tbody>
</table>

Table 4
3. Theorems.

Unfortunately it is not possible to completely classify the non-unimodal partitions \( \lambda \). In this section we will give several infinite families of partitions which are not unimodal in Theorems 3, 4, 5, 6, 8, and 9. We also give in Theorems 7 and Theorem 11 two infinite families of unimodal partitions.

From Table 2 it appears that the following theorem holds.

**Theorem 1.** The partition \( \lambda = (2k, 2k, 4, 4) \) is non-unimodal for \( k \geq 4 \) at \( i = 2k + 7 \), with consecutive differences of -1 and 1.

We do not prove Theorem 1 here, because Theorem 3 generalizes Theorem 1. Table 2 also indicates that a similar theorem should hold for \( (2k, 2k, 11, 11) \). Note that both 4 and 11 occur on Table 3 for \( a = 90 \). Then Table 3 might indicate that there is a similar theorem for 4, 11, 12, 18, 19, 20, 26, 27, 33, 34, 35, 41, 42, and 45.

For \( \lambda = (2k + 1, 2k + 1, 8, 8) \) we have the next theorem.

**Theorem 2.** The partition \( \lambda = (2k + 1, 2k + 1, 8, 8) \) is non-unimodal for \( k \geq 8 \) at \( i = 2k + 15 \), with consecutive differences of -1 and 1.

Again Table 3 indicates that a similar theorem may hold for 8, 15, 16, 22, 23, 30, 31, 37, 38, and 43.

We now come to the theorems for partitions \( \lambda = (a, a, b, b) \) which give the above two sequences of \( b \)'s, and generalize Theorems 1 and 2.

**Theorem 3.** Let \( a \) be an even integer satisfying \( a \geq (4 - \sqrt{3})b + (5 - \sqrt{3}) \). If \( b \) satisfies

1. \( \lfloor \sqrt{3}(b + 1) \rfloor \) is even, and
2. \( \sqrt{3}(b + 1)^2 + 6 \leq \lfloor \sqrt{3}(b + 1) \rfloor + 1 \leq \sqrt{3}(b + 2)^2 - 8 - 1 \),

then \( \lambda = (a, a, b, b) \) is non-unimodal at \( i = a + \lfloor \sqrt{3}(b + 1) \rfloor - 1 \). The consecutive differences are

\[
[(3b^2 + 6b - (\lfloor \sqrt{3}(b + 1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b + 1) \rfloor - 2) - 12)/12]
\]

and

\[
[(3b^2 + 12b - (\lfloor \sqrt{3}(b + 1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b + 1) \rfloor - 2))/12].
\]

**Theorem 4.** Let \( a \) be an odd integer satisfying \( a \geq (4 - \sqrt{3})b + (5 - \sqrt{3}) \). If \( b \) satisfies

1. \( \lfloor \sqrt{3}(b + 1) \rfloor \) is odd, and
2. \( \sqrt{3}(b + 1)^2 + 9 \leq \lfloor \sqrt{3}(b + 1) \rfloor + 1 \leq \sqrt{3}(b + 2)^2 - 11 - 1 \),

then \( \lambda = (a, a, b, b) \) is non-unimodal at \( i = a + \lfloor \sqrt{3}(b + 1) \rfloor - 1 \). The consecutive differences are

\[
[(3b^2 + 6b - (\lfloor \sqrt{3}(b + 1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b + 1) \rfloor - 2) - 9)/12]
\]

and

\[
[(3b^2 + 12b - (\lfloor \sqrt{3}(b + 1) \rfloor - 2)^2 - 6(\lfloor \sqrt{3}(b + 1) \rfloor - 2) - 3)/12].
\]
Proof. We prove Theorem 3. A straightforward but tedious calculation shows that
\begin{equation}
(1 - q)G(Y_{3\lambda})(q) = \frac{1}{(1 - q^2)(1 - q^3)(1 - q^4)} - \frac{q^{2a} + 3}{(1 - q)(1 - q^2)(1 - q^3)} + \frac{q^{4b} + 5}{(1 - q)(1 - q^2)(1 - q^4)} - \frac{q^{a+b} + 5}{(1 - q)(1 - q^2)(1 - q^3)} + \frac{q^{a+b+3} + 3}{(1 - q^2)(1 - q^3)} - \frac{q^{2a} + 3}{(1 - q)(1 - q^2)^2} + \frac{q^{2a+b+6}}{(1 - q)(1 - q^2)^2}.
\end{equation}

Clearly each term in (3.1) can be expanded in a Taylor series in \(q\), with coefficients of \(q^n\) which are pseudo polynomials in \(n\) \([7, \S 4.4]\). Assume for the time being that \(a \geq 4b + 5\). Then for \(n\) in the interval from \(a + 1\) to \(a + 2b + 2\), only the first four terms of (3.1) contribute. A MACSYMA run using these explicit pseudo polynomials shows that the coefficient of \(q^{a+j+1}\) is
\begin{equation}
\left\lfloor \frac{3b^2 + 6b - j^2 - 6j - 12}{12} \right\rfloor = \left\lfloor \frac{3b^2 + 6b - j^2 - 6j - 5}{12} \right\rfloor \quad \text{for } a \text{ even and } j \text{ even,}
\end{equation}
\begin{equation}
\left\lfloor \frac{3b^2 + 12b - j^2 - 6j}{12} \right\rfloor = \left\lfloor \frac{3b^2 + 12b - j^2 - 6j + 7}{12} \right\rfloor \quad \text{for } a \text{ even and } j \text{ odd,}
\end{equation}
\begin{equation}
\left\lfloor \frac{3b^2 + 12b - j^2 - 6j - 3}{12} \right\rfloor = \left\lfloor \frac{3b^2 + 12b - j^2 - 6j + 4}{12} \right\rfloor \quad \text{for } a \text{ odd and } j \text{ even,}
\end{equation}
\begin{equation}
\left\lfloor \frac{3b^2 + 6b - j^2 - 6j - 9}{12} \right\rfloor = \left\lfloor \frac{3b^2 + 6b - j^2 - 6j - 2}{12} \right\rfloor \quad \text{for } a \text{ odd and } j \text{ odd.}
\end{equation}

Some elementary algebra then implies Theorem 3 for \(a \geq 4b + 5\). This inequality on \(a\) may be relaxed to \(4b + 5 \leq a + \lceil \sqrt{3}(b + 1) \rceil\), so that the four terms of (3.1) still contribute to the two offending terms. \(\square\)

We see that the sequence of \(b\)'s for Theorem 3 (Theorem 4) does not include 45 (44) as suspected. It does appear that the allowed \(b\)'s for Theorem 3 lie in residue classes modulo 15. However this is not correct. It can be shown, for example, that \(b = 15m + 11, 0 \leq m \leq 26\) satisfies Theorem 3, but \(b = 15 \times 27 + 11\) does not. Strictly speaking, Theorem 3 (Theorem 4) with \(b = 4 (b = 8)\) implies Theorem 1 (Theorem 2) for \(k \geq 7\) \((k \geq 11)\). Nevertheless, these two theorems can be established independent of Theorems 3 and 4.

Note also that condition (1) in Theorems 3 and 4 implies that a given \(b\) may not satisfy both theorems. Because \(N\sqrt{3} - \lfloor N\sqrt{3} \rfloor\) is equidistributed on \([0,1)\) ([5, Prob. 166]), it can be shown that the density of the \(b\)'s satisfying Theorem 3 or 4 is \((\sqrt{3} - 1)/2\).

It is also clear that the bound for \(a\) in Theorems 3 and 4 is not the best possible, for example one might conjecture that \(a \geq 2b\) is sufficient. However, \(b = 12\) is allowed by Theorem 3 and \((24, 24, 12, 12)\) is unimodal. (It does not appear on
Moreover, if $a > 2b$ is not sufficient, for $b = 35$, $a \geq 78$. It is possible to give a general theorem in the range $2b + 2 \leq a + 1 \leq 3b + 3$, but the inequalities are not as nice as condition (2) in Theorem 3. For the range $3b/2 + 1 \leq a + 1 \leq 2b + 1$, for example $(11, 11, 6, 6)$, there is another simple sufficient condition, which we state in the next two theorems.

**Theorem 5.** If $k \geq 2$ and $2 \leq t \leq (1 + \sqrt{1 + 24k})/4$, then the partition $\lambda = (3k+t, 3k+t, 2k, 2k)$ is non-unimodal at $i = 6k + 2t - 1$. The consecutive differences are

$$- \left\lfloor \frac{t^2 - t}{3} \right\rfloor$$

and

$$\left\lfloor \frac{3k - 2t^2 + t + 6}{6} \right\rfloor.$$

**Theorem 6.** If $k \geq 2$ and $2 \leq t \leq (1 + \sqrt{1 + 24k})/4$, then the partition $\lambda = (3k + t + 2, 3k + t + 2, 2k + 1, 2k + 1)$ is non-unimodal at $i = 6k + 2t + 3$. The consecutive differences are

$$- \left\lfloor \frac{t^2 - 1}{3} \right\rfloor$$

and

$$\left\lfloor \frac{3k - 2t^2 - t + 6}{6} \right\rfloor.$$

**Proof.** This time three terms of (3.1) contribute to the coefficient of $q^{3b+3+j}$, for $0 \leq j \leq a - b$. The terms given in Theorems 5 and 6 are the differences given by MACSYMA, and the inequality on $t$ insures that the differences are negative and positive. □

Next we see that Table 1 lists partitions with four or six parts, which suggests that a partition with at most three parts is unimodal. This is true, and we will give a proof similar to the proof of Theorems 5 and 6. However the computations can be simplified by using the following lemma.

**Lemma 1.** For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, we have

$$(1 - q)G(Y_\lambda)(q) = G(Z)(q) - q^{\lambda_1 + 1}G(Y_\mu)(q),$$

where $Z$ is the set of all partitions inside $\lambda$ whose first two parts are equal, and $\mu$ is the partition $(\lambda_2, \lambda_3, \ldots, \lambda_k)$.

**Proof.** Let $A = \{\emptyset, 1\}$ and consider the set $Y \times A$ whose generating function is $(1 - q)G(Y_\lambda)(q)$ if the sign of $1 \in A$ is $-1$. A sign-reversing involution $\sigma$ on $Y \times A$ is given by $\sigma((\gamma, \emptyset)) = (\mu, 1)$, where $\mu = (\gamma_1 - 1, \gamma_2, \ldots, \gamma_k)$ if $\gamma_1 > \gamma_2$; and $\sigma((\gamma, 1)) = (\mu, \emptyset)$, where $\mu = (\gamma_1 + 1, \gamma_2, \ldots, \gamma_k)$ if $\gamma_1 < \lambda_1$. Clearly the fixed points of $\sigma$ have $\gamma_1 = \gamma_2$ or $\gamma_1 = \lambda_1$, whose generating function is given in Lemma 1. □
Proposition 1. If $\lambda = (a, b, c)$, then
\begin{equation}
(1 - q)G(Y_\lambda)(q) = \frac{1}{(1 - q^2)(1 - q^3)} - \frac{q^{3a+3}}{(1 - q^2)(1 - q^3)} - \frac{q^{2b+2}}{(1 - q)(1 - q^2)} + \frac{q^{2b+c+3}}{(1 - q)(1 - q^2)} - \frac{q^{a+1}}{(1 - q)(1 - q^2)} + \frac{q^{a+b+2}}{(1 - q)^2} - \frac{q^{a+b+c+3}}{(1 - q)^2} - \frac{q^{a+2c+3}}{(1 - q)^2}.
\end{equation}

Proof. An easy calculation shows that Lemma 1 implies Corollary 1, where the first four terms of (3.3) are $G(Z)(q)$ and the last four terms are $-q^{a+1}G(Y_\mu)(q)$. □

Theorem 7. If $\lambda$ has at most three parts, then $\lambda$ is unimodal.

Proof. We indicate the proof if $\lambda$ has three parts. From Lemma 1, we see that $(1 - q)G(Y_\lambda)(q)$ is the difference of two terms which are given explicitly in Proposition 1. If each term were unimodal, we could conclude in this case that $\lambda$ is unimodal. Unfortunately, this is not true, but a careful case-by-case analysis shows that $\lambda$ is unimodal. □

The next observation is that the non-unimodal $\lambda$ in Table 1 lie in intervals. For example, $(12, 10, 4, 4)$, $(12, 11, 4, 4)$ and $(12, 12, 4, 4)$ are all non-unimodal at $i = 21$ with the same three values of $a_i$, and they form the interval $[(12, 10, 4, 4), (12, 12, 4, 4)]$. The reason is clear: if a cell in position $(j + 1, k + 1)$ is removed from the Ferrers diagram of $\lambda$, the coefficients of $q^n$ in $G(Y_\lambda)(q)$ do not change for $0 \leq n \leq jk + j + k$. Thus if $j$ and $k$ are chosen so that $jk + j + k \geq i + 1$, then $\lambda$ with the cell $(j + 1, k + 1)$ removed will also be non-unimodal. For example, we see that Theorem 1 implies that $(2k, m, 4, 4)$ is non-unimodal for $m \geq k + 4$. It is possible to state a general theorem corresponding to Theorems 3 and 4, instead we give such a theorem for Theorems 5 and 6.

Theorem 8. Let $2 \leq t \leq (1 + \sqrt{1 + 24k})/4$. Any partition in the following intervals is non-unimodal:

1. $[(3k + t, 3k + t, 2k, 2k - [(2k + 3 - 2t)/4]), (3k + t, 3k + t, 2k, 2k)]$ or
2. $[(3k + t + 2, 3k + t + 2, 2k + 1, 2k + 1 - [(2k - 1 - 2t)/4]), (3k + t + 2, 3k + t + 2, 2k + 1, 2k + 1)]$.

By considering the non-unimodal partitions of $n \leq 50$, two more infinite families, each singly indexed, can be found: $(k + 2, k, k, k)$, for $k = 10$ or $k \geq 12$, non-unimodal at $i = 2k + 3$; and $(2k + 4, 2k + 4, 2k + 4, 2k + 4, 2k + 2)$ for $k \geq 4$, at $i = 4k + 7$. In fact, the cases $(a, a, a, b)$ and $(a, b, b, b)$ could be done just as $(a, a, b, b)$ was, but we shall be content to give these two families. In the first case cells from two different rows may be deleted to create non-unimodal intervals.

Theorem 9. Any partition in the following intervals is non-unimodal:

1. $[(k + 2, k, [(2k + 2)/3], [(2k + 1)/4]), (k + 2, k, k, k)]$ for $k = 10$ or $k \geq 12$, or
2. $[(2k + 4, 2k + 4, [(4k + 5)/3], k), (2k + 4, 2k + 4, 2k + 4, 2k + 4, k)]$ for $k \geq 4$.

The respective consecutive differences are

1. $-1$ and $[k/6] - 1$ for $k \not\equiv 4 \pmod{6}$; and $-1$ and $[k/6]$ for $k \equiv 4 \pmod{6}$, and
2. $-1$ and $[(k + 1)/3] - 1$ for $k \not\equiv 1 \pmod{3}$; and $-1$ and $[(k + 1)/3]$ for $k \equiv 1 \pmod{3}$.
Proof. First we verify the non-unimodality claim for \((k + 2, k, k, k)\). This follows from
\[
G(\mathcal{Y}_\lambda)(q) = \left[\frac{k + 4}{4}\right]_q + \left(q^{k+1} + q^{k+2}\right)\left[\frac{k + 3}{3}\right]_q
\]
and some lengthy calculations involving the appropriate pseudo polynomials. The second part is verified by noting that \((2k+4, 2k+4, 2k+4, k)\) and \((2k+4, 2k+2, 2k+2, 2k+2)\) contain the same partitions of \(i\) for \(i \leq 4k + 3\). For \(i = 4k+6, 4k+7,\) and \(4k+8\) respectively, \((2k+4, 2k+4, 2k+4, k)\) contains \(1, 2,\) and \(4\) partitions that \((2k+4, 2k+2, 2k+2, 2k+2)\) does not contain. Similarly for \(i = 4k+6, 4k+7,\) and \(4k+8\), \((2k+4, 2k+2, 2k+2, 2k+2)\) contains \(2, 3,\) and \(5\) partitions that \((2k+4, 2k+4, 2k+4, k)\) does not contain. Thus the consecutive differences are the same at \(i = 4k+7\) and \(i = 4k+8\), which establishes (2). \(\square\)

How many non-unimodal partitions of \(n\) are there? Table 4 and Theorem 8 imply that these numbers are non-zero for \(n \geq 30\). The intervals of Theorem 8 or Theorem 9 imply the following theorem. It is very likely, however, that this number grows much more rapidly than Theorem 10 asserts.

**Theorem 10.** As \(n \to \infty\), the number of non-unimodal partitions of \(n\) is at least \(cn^2\).

We also see from Table 4 that the number of non-unimodal partitions of \(n\) is even for \(n \leq 50\). In view of (1.2), this could suggest that self-conjugate partitions are unimodal. In fact, no self-conjugate partition appears on the list of all non-unimodal partitions of \(n \leq 50\). Moreover, all self-conjugate partitions of \(n \leq 124\) are unimodal. The following theorem is a partial result in this direction.

**Theorem 11.** If \(\lambda\) is any self-conjugate partition whose Durfee square has size at most two, then \(\lambda\) is unimodal.

Proof. We may assume that the Durfee square of \(\lambda\) has size two, \(\lambda = (a + 2, b + 2, 2b, 1^{a-b})\), where \(b \leq a\). If we apply Lemma 1 to \(\lambda\) we find

\[
G(\mathcal{Z})(q) = 1 + (q^2 + \ldots + q^{a+2}) + \frac{q^4}{(1-q)(1-q^2)^2} - \frac{q^{a+5}}{(1-q)^2(1-q^2)} + \frac{q^{a+b+6}}{q^{a+3b+8}}
\]

and

\[
G(\mathcal{Y}_\mu)(q) = (1 + q + \ldots + q^{a+1}) + \frac{q^2}{(1-q)^3(1-q^2)} - \frac{q^{a+3}}{(1-q)^3} + \frac{2q^{a+b+4}}{(1-q)^3}
\]

Again a case-by-case analysis implies Theorem 10. (The case \(b \leq a \leq 2b\) is particularly unpleasant.) \(\square\)
4. Remarks. There are several observations which can be made that have not led to theorems. The purpose of this section is to comment on these possible theorems.

Observation 1. All examples of non-unimodal partitions are bimodal.

Observation 2. All examples of non-unimodal partitions are non-unimodal at an odd integer $i$.

Observation 3. All examples of non-unimodal partitions have their absolute peaks at $i - 1$ or $i + 1$ if they are non-unimodal at $i$.

It would appear very unlikely that Observations 1-3 are theorems, rather they are properties of the infinite families that have been found so far.

Observation 4. There are no examples of non-unimodal partitions with 5, 7, or 9 parts.

This has been checked for 5 parts with part size $\leq 30$, 7 parts with part size $\leq 15$, and 9 parts with part size $\leq 10$. Again it appears that there is just not enough data in this case.

Observation 5. All examples of infinite families of non-unimodal partitions have four parts. The only examples of non-unimodal partitions with six parts lie in intervals associated with $(10, 9, 9, 9, 9, 9)$, $(8, 8, 8, 8, 8, 2)$, or $(8, 8, 6, 6, 6, 6)$.

It is remarkable that $(10, 9, 9, 9, 9, 9)$ is non-unimodal, being so close to $(9, 9, 9, 9, 9, 9)$, which is unimodal. These three examples have resisted all attempts to be placed in an infinite family.

Observation 6. The probability that a partition of $n$ is non-unimodal roughly decreases to 0.00014 at $n = 52$.

The word “roughly” is used because the probability is not strictly decreasing. For $42 \leq n \leq 52$ the probability lies between 0.00014 and 0.00030. (The last integer for which it has been computed is $n = 52$.) One might conjecture that the probability $\to 0$ as $n \to \infty$.

Conjecture 1. All self-conjugate partitions are unimodal.

Conjecture 1 has been verified for all self-conjugate partitions of $n \leq 124$. (There are 174181 such partitions). It is also supported by Theorem 11.

Conjecture 2. The staircase partition $\lambda = (n, n - 1, \ldots, 1)$ is unimodal.

Conjecture 2 has been verified for $n \leq 22$. The generating function was considered by Carlitz [2]. It is also related to the Rogers-Ramanujan continued fraction [4, §19.15]. If $G_n(\mathcal{Y}_\lambda)(q)$ is the generating function for $\lambda = (n - 1, n - 2, \ldots, 1)$, and $G_0(\mathcal{Y}_\lambda)(q) = 1$, it is well-known [3] that $G_n(\mathcal{Y}_\lambda)(q)$ is $q$-analogue of the $n$th
Catalan number. It is not hard to see that

\[
(4.1) \quad \sum_{n=0}^{\infty} G_n(\mathcal{Y}_\lambda)(1/q)q^{(n-1)/2}x^n = \frac{1}{1 - \frac{x}{1 - \frac{xq}{1 - \frac{xq^2}{1 - \dotsc}}}}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-x)^n q^{n^2}}{(q)_n} / \sum_{n=0}^{\infty} \frac{(-x)^n q^{n^2-n}}{(q)_n},
\]

where

\[
(q)_n = \prod_{k=1}^{n} (1 - q^k).
\]

Thus, Conjecture 2 is equivalent to a unimodality property of the continued fraction in (4.1).

Several other questions about Young’s lattice remain open. The existence of a symmetric chain decomposition for a \(m \times n\) rectangle, \(m \geq 5\) is open. Clearly the rectangles are the only partitions which are symmetric. What happens if skew shapes are allowed? It is also known that Young’s lattice of a rectangle has the Sperner property [8]. Susanna Fishel and the author have shown that the Young’s lattice of any partition of \(n \leq 26\) has the Sperner property. Finally, it is clear that one would not have found the infinite families of non-unimodal partitions without aid of a computer. What is missing is an algebraic formulation for a general partition \(\lambda\) (see [6] and [8]).

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**References**