1. The characteristic polynomial of the homogeneous equation \( u'' + u' + u = 0 \) is \( \lambda^2 + \lambda + 1 = 0 \). The roots are \( \lambda_{1,2} = \frac{-1 \pm \sqrt{3}i}{2} = e^{\pm \frac{\sqrt{3}}{2}i} \). The general solution of the homogeneous equation is \( C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \). (There are other equivalent expressions, such as \( [c_1 e^{-\frac{\sqrt{3}}{2}t} \cos \left( \frac{\sqrt{3}}{2}t \right) + c_2 e^{-\frac{\sqrt{3}}{2}t} \sin \left( \frac{\sqrt{3}}{2}t \right)] \) or \( C e^{-\frac{\sqrt{3}}{2}t} \cos(\frac{\sqrt{3}}{2}(t - t_0)) \).)

We need to find a particular solution for the inhomogeneous equation. As \( 3 \sin(\sigma t) = 3 \text{Im} e^{\sigma t} \), we can first solve \( u'' + u' + u = 3e^{\sigma t} \) and then take the imaginary part. As we did in class, we seek the solution of the last equation as \( A e^{\sigma t} \). This gives \( A = \frac{3}{1 - \sigma^2 + \sigma^2} = \frac{-3\sigma}{(1 - \sigma^2)^2 + \sigma^2} \cos \sigma t + \frac{3(1 - \sigma^2)}{(1 - \sigma^2)^2 + \sigma^2} \sin \sigma t \). The general solution of the inhomogeneous equation then is \( u(t) = v(t) + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \).

(This expression can again be written in several ways.) One can also find the particular solution of the inhomogeneous equation by starting from \( a \cos \sigma t + b \sin \sigma t \). When we substitute this expression into the equation, we get a system of two equations for the two unknowns \( a, b \), which we can solve and arrive at \( a = \frac{-3\sigma}{(1 - \sigma^2)^2 + \sigma^2}, \quad b = \frac{3(1 - \sigma^2)}{(1 - \sigma^2)^2 + \sigma^2} \), confirming the previous calculation.

2. We need to maximize \(|A|\) from the previous problem. This is the same as minimizing \((1 - \sigma^2)^2 + \sigma^2\). Setting \( \sigma^2 = \tau \), we need to minimize \( g(\tau) = (1 - \tau)^2 + \tau \) over \( \tau \geq 0 \). We can write \( g(\tau) = (\frac{1}{4} - \tau)^2 + \frac{1}{4} \) from which we see that the minimum is attained at \( \tau = \frac{1}{2} \). (Instead of completing the square, we can work with the equation \( g'(\tau) = 0 \).) Going back to \( \sigma \) we obtain \( \sigma = \pm \frac{\sqrt{2}}{2} \). If we work in the real setting, writing the solution in the form \( a \cos \sigma t + b \sin \sigma t \), we need to use the fact the the amplitude of the function given by the last expression is \( \sqrt{a^2 + b^2} \). (This can be seen several ways, for example by writing \( a \cos \sigma t + b \sin \sigma t = \text{Re}(a - ib)e^{\sigma t}, \) or \( a \cos \sigma t + b \sin \sigma t = \sqrt{a^2 + b^2} \cos(\sigma t + s) \) for a suitable \( s \).

3. We will solve \( u'' + x = e^{\sigma t} \) and take the imaginary part. The general solution of the homogeneous equation is \( x(t) = C_1 e^{\sigma t} + C_2 e^{-\sigma t} \). To calculate a particular solution of the inhomogeneous equation, we can use the variation of constant, see lecture 10 in the lecture log. In the last expression we consider \( C_1 \) and \( C_2 \) as functions of \( t \) and set \( C_1 e^{\sigma t} + C_2 e^{-\sigma t} = 0 \). The inhomogeneous equation then gives \( C_1' e^{\sigma t} - iC_2' e^{-\sigma t} = e^{\sigma t} \). Solving for \( C_1', C_2' \) (by using Cramer’s rule, for example), we obtain \( C_1' = -\frac{1}{2}, \quad C_2' = \frac{1}{2} \). Hence we can take \( C_1 = -\frac{t}{2}, \quad C_2 = \frac{1}{2} e^{2it} \). Then \( C_1 e^{\sigma t} + C_2 e^{-\sigma t} = e^{\sigma t}(-\frac{1}{2} + \frac{1}{2}) \). Noticing that \( e^{\sigma t} \) is a solution of the homogeneous equation, we can take for our particular solution the function \( -\frac{t}{2} e^{\sigma t} \). To obtain a particular solution of \( x'' + x = \sin t \), we take the imaginary part of \( -\frac{t}{2} e^{\sigma t} \), obtaining \(-\frac{1}{2} t \cos t \). One can check directly that this is a particular solution of our equation. The general solution then is \( x(t) = -\frac{1}{2} t \cos t + C_1 e^{\sigma t} + C_2 e^{-\sigma t} \) here \( C_j \) are now constants, or, alternatively, \( x(t) = -\frac{1}{2} t \cos t + c_1 \cos t + c_2 \sin t \), where \( c_1, c_2 \) are again constants. One can
also do the variation of constants starting from \( c_1 \cos t + c_2 \sin t \), considering \( c_1, c_2 \) as functions of \( t \). If you do it this way, you may obtain expressions such as, for example, \( x(t) = -\frac{1}{2} t \cos t + \frac{1}{2} \sin 2t \cos t + \frac{1}{2} \sin^3 t \). This may at first look different than the expression obtained above, but it describes the same solutions: we note that \( \frac{1}{2} \sin 2t \cos t + \frac{1}{2} \sin^3 t = \frac{1}{2} \sin t \cos^2 t + \frac{1}{2} \sin t \sin^2 t = \frac{1}{2} \sin t \) and the last function solves the homogeneous equation.

4. We have \((t')' = rt^{-1}\) and \((t')'' = r(r-1)t^{-2}\). Substituting these expression into the equation, we get \(ar(r-1) + br + c = 0\). Alternatively, we can use the substitution \( t = e^s\). Our equation then changes to \( ax'' + (b-a)x' + cx = 0\) and the function \( t''\) changes to \( e^{s''}\). The characteristic equation for \( r \) will now be \( ar^2 + (b-a)r + c = 0\), which is the same as \( ar(r-1) + br + c = 0\).

5. The linear space of the solutions of the homogeneous equation has dimension 2 in this case. Hence we only have to show that the functions \( t^{r_1} \) and \( t^{r_2} \) are linearly independent over \( C \) in \((0,\infty)\). Let us consider the equation \( C_1 t^{r_1} + C_2 t^{r_2} = 0\) for some constants \( C_1, C_2\). Assuming the equation is satisfied at \( t = t_1 > 0\) and at \( t = t_2 > 0\), \( t_2 \neq t_1\), we see that the constants \( C_1, C_2\) must vanish when \( \det \left( \begin{array}{cc} t_1^{r_1} & t_1^{r_2} \\ t_2^{r_1} & t_2^{r_2} \end{array} \right) = 0 \). Letting \( \frac{t_2}{t_1} = s\), we see that the determinant will not vanish when \( s^{r_1} \neq s^{r_2}\), which is the case as long as \( s \neq 1 \) and \( r_1 \neq r_2\). Hence when \( r_1 \neq r_2\) the the expression \( C_1 t^{r_1} + C_2 t^{r_2}\) is a general solution. Alternatively, we can use the change of variables \( t = e^s\) to reduce our example to the case of the equation with the constant coefficients.

6. We have \( \frac{d}{dx} E(t) = m \ddot{x} + V'(x) \dot{x} = \dot{x}(m \ddot{x} + V'(x)) = -\alpha \dot{x}^2 \leq 0\).

7*. (Optional) Substituting \( p(z) = C \rho(z) \) into the equation \( \frac{dp}{dz} = -g(z) \rho(z)\), we obtain \( \frac{dp}{dz} = -g(z) \rho(z) \frac{\dot{z}}{z} \), which is the same as \( \frac{dp}{\dot{z}} = -\frac{g(z)}{z} dz \). Integrating between \( \rho_0 \) and \( \rho \) on the left-hand side and between 0 and \( z \) on the right-hand side, we obtain \( \frac{\rho}{\rho_0} = -\frac{1}{z} (V(z) - V(0)) \), where \( V(z) = -\frac{g(z)z}{(h+z)} \). This gives \( \rho = \rho_0 e^{-\frac{V(z)-V(0)}{c}} \). Then \( \lim_{z \to \infty} \rho(z) = \rho_0 e^{-\frac{V(0)}{c}} > 0\), and hence the mass of the atmosphere cannot be finite (assuming the atmosphere is at equilibrium). When \( g \) is constant, a similar (an, in fact, easier) calculation gives \( \rho = \rho_0 e^{-\frac{z}{c^2}} \), which is equivalent to replacing \( V(z) - V(0) \) by \( V'(0)z \) in the formula for variable \( g \).

8*. (Optional) We have \( x'(t) = p(x(t)) \). Hence \( x'' = \frac{dp}{dx} x' = p \frac{dp}{dx} \). Hence \( x'' = f(x, x') \) gives \( p \frac{dp}{dx} = f(x, p) \).

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1 Other forms are possible, depending on how we choose the constants of integration.