1. Consider the second order equation with (smooth) variable coefficients

\[ x'' + p(t)x' + q(t)x = 0 \quad (1) \]

in an interval \( I = (t_1, t_2) \). Assume that \( x_1, x_2 \) are two solutions of (1) and set

\[ w = x_1x'_2 - x_2x'_1. \quad (2) \]

The function \( w \) is called the Wronskian of the two solutions \( x_1, x_2 \). Show that

(i) \( w \) satisfies the equation \( w' + p(t)w = 0 \);

(ii) if \( w(t_0) \neq 0 \) for some \( t_0 \in I \), then \( w(t) \neq 0 \) for every \( t \in I \).

2. Find the general solution of the equation

\[ x'' + \frac{x'}{t} = 1 \quad (3) \]

on the interval \( (0, \infty) \). (Hint: First solve the homogeneous equation and then either use the variation of constants or try to guess a particular solution.)

3. Consider the equation

\[ x'' + \frac{2x'}{t} + x = 0 \quad (4) \]

in \( (0, \infty) \). Show that the substitution \( x = \frac{z}{t} \) changes our equation with variable coefficients to an equation with constant coefficients and find a basis of the space of solutions of (4).

4.∗ (Optional) Exercises 1-3 on page 190.

5.∗ (Optional) For an \( \mathbb{R}^2 \)-valued function \( x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) consider the system

\[ x'' + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x' + x = 0. \quad (5) \]

Find all bounded solutions of the system in the interval \( (0, \infty) \).

Hint: one can either write our equation as a first order \( 4 \times 4 \) system (by setting \( x' = y \) and writing down the equation satisfied by \( y \) or one can search directly for solutions of the form \( be^{\lambda t} \) in a way similar to what we have done for first-order systems.