1. Let \( G_{B_R}(x, y) = -\frac{1}{4\pi|x-y|} + \frac{R}{4\pi|x-y|^3} \) (where \( y^* = y \frac{R^2}{|y|^2} \)) be the Green’s function for \( B_R \). Let us denote \( y' = (y_1, y_2, -y_3) \). For \( y \in B_R^+ \) we set \( G(x, y) = G_{B_R}(x, y) - G_{B_R}(x, y') \). Then clearly \( G(x, y) \) vanishes when \( |x| = R \). When \( x_3 = 0 \), then \( G_{B_R}(x, y) = G_{B_R}(x, y') \), because in that case we have \( |x-y| = |x-y'| \) and \( |x-y^*| = |x-y'|^* \), due to the reflection symmetry about the \( x_1x_2 \) plane. Hence \( G(x, y) \) vanishes when \( x_3 = 0 \) and we see that the function \( x \mapsto G(x, y) \) vanishes at the boundary of \( B_R^+ \). Clearly \( \Delta G(x, y) = \delta(x-y) \) in \( B_R^+ \) (as \( y \) is the only point of the set \( y, y^*, y', y'' \) which lies in \( B_R^+ \)), and therefore \( G(x, y) \) is the desired Green’s function of \( B_R^+ \).

2. Extend \( f \) first to the square \([0,1] \times [0,1]\) by \( f(x_1, x_2) = -f(x_2, x_1) \). Then to the square \([0,2] \times [0,2]\) by \( f(x_1, x_2) = -f(2-x_1, x_2); (x_1, x_2) \in [1,2] \times [0,1], f(x_1, x_2) = f(2-x_1, 2-x_2), (x_1, x_2) \in [0,1] \times [1,2]; f(x, y) = f(2-x, y) = f(x, 2-y), f(x_1, x_2) = f(2-x_1, 2-x_2), (x_1, x_2) \in [1,2] \times [1,2] \). And finally to a function on \( \mathbb{R}^2 \) (still denoted by \( f \)) which is \( 2 \)-periodic in \( x_1 \) and \( 2 \)-periodic in \( x_2 \). Set \( F(x) = f(2x) \). The function \( F \) is \( 1 \)-periodic in \( x_1 \) and \( x_2 \), and we use the machine to calculate its Fourier coefficients \( \hat{F}(k) \). We set \( \hat{U}(k) = -\frac{f(k)}{\pi|k|^2} \) (note that there is no \( 4 \) in front of \( \pi^2 \)) when \( k \neq 0 \) and \( \hat{U}(0) = 0 \), and use the machine to obtain \( U(x) = \sum_k \hat{U}(k)e^{2\pi ikx_1x_1+kx_2x_2} \). The function \( U \) solves \( \frac{1}{4} \Delta U = F \), with the factor \( \frac{1}{4} \) coming from skipping the \( 4 \) in front of \( \pi^2 \) as noted above. We set \( u(x) = U(x) \) and note that \( \Delta u(x) = \frac{1}{4} \Delta U(x) = F(x) \). The function \( u \) has the same symmetries as \( f \), and hence vanishes at the boundary of our triangle, and is the (unique) solution to our problem.

3. Searching \( u(x) = \frac{f(r)}{r} \) (with \( r = |x| \)), we obtain the equation \( \nu'' = -\lambda \nu \) for \( \nu \). We are looking for solutions which vanish at \( r = 1 \). In addition, the solutions also have to vanish at \( r = 0 \), so that \( u(x) = \frac{f(r)}{r} \) is not singular at \( r = 0 \). We have seen this problem before. The solutions are of the form \( \nu(r) = \sin(\pi kr), \lambda = \pi^2 k^2, k = 1, 2, \ldots \) Hence the radial eigenfunctions are \( \frac{\sin \pi kr}{r} \) with the corresponding eigenvalues \( \pi^2 k^2 \).

4. It is enough to prove the statement for \( P(z) = z^m \). We have \( \frac{\partial}{\partial z} (x+i y)^m = m(x+iy)^{m-1} \), \( \frac{\partial^2}{\partial z^2} (x+i y)^m = m(m-1)(x+iy)^{m-2} \). This can also be written as \( \frac{\partial^2}{\partial x^2} z^m = m(m-1)z^{m-2} \). Similarly, \( \frac{\partial^2}{\partial y^2} (x+i y)^m = im(x+iy)^{m-1} \), \( \frac{\partial^2}{\partial x \partial y} (x+i y)^m = -m(m-1)(x+iy)^{m-2} \). Hence \( \Delta z^m = 0 \). There are many other ways to arrive at the same conclusion. For example, one can use the polar coordinates to write \( z^m = r^m e^{im\theta} \) and use the expression \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \) to check that \( r^m e^{im\theta} \) is harmonic.

Remark: The above problem asks to verify “by hand” a statement which is arises from some basic considerations of complex analysis. There one works with \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \) and \( \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), \) with \( i = \sqrt{-1} \). One can check by an easier version of the above calculations that \( \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \Delta \).

5. Option 1: The direction perpendicular to \( H \) is given by the vector \( a = (1,1,1) \). Hence for \( x \in \mathbb{R}^3 \), its projection \( x' = Px \) will be given by the conditions \( x' = x - ta \) and \( x_1' + x_2' + x_3' = 0 \). The last condition is the same as \( x_1 + x_2 + x_3 = 3t = 0 \), which gives \( t = \frac{x_1 + x_2 + x_3}{3} \). Substituting this into the expression \( x' = x - ta \), we obtain \( P = \left( \begin{array}{ccc} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{array} \right) \).

Option 2: We need to find two mutually perpendicular vectors \( a, b \in H \). For our particular \( H \) it is not hard to find such vectors without much calculation. For example, \( a = (1,1,-1,-1) \) and \( b = (1,-1,1,1) \) have the desired properties.

The length of both \( a \) and \( b \) is \( \sqrt{1+1+1+1} = 2 \), so the desired projection is \( P = \frac{1}{4} a \otimes a + \frac{1}{4} b \otimes b = \left( \begin{array}{cccc} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array} \right) \).

6. For \( u(x) = (R^2 - x_1^2 - x_2^2 - x_3^2)(b_1 x_1 + b_2 x_2 + b_3 x_3 + b_0) \), we obtain \( \Delta u = -10 b_1 x_1 - 10 b_2 x_2 - 10 b_3 x_3 - 6 b_0 \). Hence the solution is obtained by taking \( b_j = -\frac{1}{10}, j = 1, 2, 3 \) and \( b_0 = -\frac{1}{6} \).

In general, if \( P \) is a polynomial of degree at most \( m \) in \( x_1, x_2, x_3 \), then \( Lu = \Delta [(R^2 - x_1^2 - x_2^2 - x_3^2) P(x_1, x_2, x_3)] \) is again a polynomial of degree at most \( m \). Denoting by \( P_m \) the linear space of all polynomials of degree \( \leq m \) in \( \mathbb{R}^3 \), we see that \( L \) maps \( P_m \) to \( P_m \). Clearly \( L \) is a linear mapping (and hence it can be represented by a matrix, if we choose a bases in \( P_m \). We claim that the equation \( LP = 0 \) for \( P \in P_m \) only has the trivial solution \( P = 0 \). To see that, we note that \( LP = 0 \) implies that \( u = (R^2 - |x|^2)P \) is harmonic. At the same time, \( u \) vanishes at the boundary of \( B_R \). These two facts imply that \( u = 0 \) and hence also \( P = 0 \). (One can use the maximum principle, for example, or other methods used to prove uniqueness of solutions for \( \Delta u = 0 \) in \( B_R \) and \( \partial_{B_R} u = 0 \).) We see that the equation \( LP = 0 \) for \( P \in P_m \) has only the trivial solution \( P = 0 \). Hence the equation \( LP = Q \) has a unique solution \( P \in P_m \) for every \( Q \in P_m \).