Do at least four of the following six problems.\footnote{For grading purposes, any 4 problems correspond to 100%. You can get extra credit if you do more.}

1. Consider the half-ball \( B_R^+ \) of radius \( R \) in \( \mathbb{R}^3 \) given by

\[
B_R^+ = \{ x = (x_1, x_2, x_3), \ x_1^2 + x_2^2 + x_3^3 < R^2, \ x_3 > 0 \}. \tag{1}
\]

Find a formula for the Green’s function of \( B_R^+ \) using the method of images.

Hint: Use the Green’s function of the ball \( B_R \) (see, for example, formula (121) in the Lecture Log) as a starting point.

2. Let \( \Omega \subset \mathbb{R}^2 \) be the interior of the triangle given by the points \((0,0), (0,1)\) and \((1,1)\). For functions \( f: \Omega \to \mathbb{R} \) we wish to solve the problem

\[
\begin{align*}
\Delta u &= f & \text{in } \Omega, \\
u &= 0 & \text{at } \partial \Omega. 
\end{align*} \tag{2}
\]

Assume that you have a machine which can do the following two tasks:

(i) For a function \( f: \mathbb{R}^2 \to \mathbb{R} \) which is \( 1 \)-periodic in both variables, the machine can calculate the Fourier coefficients \( \hat{f}(k) = \int_0^1 \int_0^1 f(x_1, x_2) e^{-2\pi i (k_1 x_1 + k_2 x_2)} \, dx_1 \, dx_2 \), \( k = (k_1, k_2) \in \mathbb{Z}^2 \).

(ii) For each set of Fourier coefficients \( \hat{f}(k), k \in \mathbb{Z}^2 \), the machine can sum the corresponding Fourier series and obtain the function \( f(x) = \sum_{k \in \mathbb{Z}^2} \hat{f}(k) e^{2\pi i (k_1 x_1 + k_2 x_2)} \).

Given \( x \in \Omega \), how can one use the machine to calculate solution of the problem (2)?

Hint: You can extend \( f \) from the triangle to a \( 2 \)-periodic function \( F \) in \( x_1 \) and \( x_2 \), so that the \( 2 \)-periodic solution of \(-\Delta u = F \) (which can be calculated using Fourier series) will satisfy \( U = 0 \) at \( \partial \Omega \). Note that the function \( x \to F(2x) \) will be \( 1 \)-periodic.

3. Let \( B_1 = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3, \ x_1^2 + x_2^2 + x_3^2 < 1 \} \) be the unit ball in \( \mathbb{R}^3 \) centered at the origin. Describe all \textit{radial} solutions of the eigenvalue problem

\[
\begin{align*}
-\Delta u &= \lambda u & \text{in } B_1, \\
u &= 0 & \text{at } \partial B_1, 
\end{align*} \tag{3}
\]

and the corresponding eigenvalues \( \lambda \).

Recall that a function \( u: B_1 \) is radial if it only depends on \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \).

Hint: Search for the solutions in the form \( u(x) = \frac{v(r)}{r} \) and note that from the first equation of (3) we get a simple equations for \( v(r) \).

4. Let \( z = x_1 + ix_2 \), and let \( P(z) \) be a polynomial in \( z \), i.e., an expression of the form \( a_0 z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m \), where \( a_0, a_1, \ldots, a_m \) are complex numbers. We can consider \( P \) as a complex-valued function of two variables \( x_1 \) and \( x_2 \) and write it as

\[
P(z) = P(x_1 + ix_2) = u(x_1, x_2) + iv(x_1, x_2), \tag{4}
\]

where \( u, v \) are real-valued functions of \( x_1 \) and \( x_2 \). Show that both \( u \) and \( v \) are harmonic, in the sense that \( \Delta u = 0 \) and \( \Delta v = 0 \).

Hint: Note that it is enough to show the statement for the special cases \( P(z) = z^m \), with \( m = 0, 1, 2, \ldots \).

5. This problem has two options. If your background in linear algebra is not very strong, choose Option 1. If you have a good background in linear algebra, choose Option 2.

Option 1: Let \( H \subset \mathbb{R}^3 \) be the linear subspace given by \( H = \{ x \in \mathbb{R}^3, \ x_1 + x_2 + x_3 = 0 \} \) and let \( P: \mathbb{R}^3 \to H \) be the orthogonal projection. Recall that \( P \) is a linear map from \( \mathbb{R}^3 \) into \( H \subset \mathbb{R}^3 \), and as such is given by a \( 3 \times 3 \) matrix. Find this matrix.

Option 2: Let \( H \subset \mathbb{R}^4 \) be the linear subspace given by \( H = \{ x \in \mathbb{R}^4, \ x_1 + x_2 + x_3 + x_4 = 0, \ x_1 - x_2 + x_3 - x_4 = 0 \} \). Let \( P: \mathbb{R}^4 \to H \) be the orthogonal projection. Find the \( 4 \times 4 \) matrix representing \( P \).

Hint: One way which works for both options is the following. Find two unit vectors \( e, f \in H \) which are perpendicular to each other. Then \( P = e \otimes e + f \otimes f \). The notation is the same as the one we used in class: for \( b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \) we define \( b \otimes b \) as the matrix with entries \( b_i b_j \). For Option 1 there is a simpler way: take \( x \in \mathbb{R}^3 \) and find \( t \in \mathbb{R} \) such that \( x = t(1,1,1) \in H \). Then \( P x = x - t(1,1,1) \).

Remark: The relevance of this problem for PDE is the following: if we have some set of functions \( \phi_1, \phi_2, \ldots, \phi_n \), and denote \( H \) the linear space of functions of the form \( c_1 \phi_1 + c_2 \phi_2 + \cdots + c_n \phi_n \), it is often useful to have an “orthogonal projection” \( P \) of a suitable class of function on the space \( H \). For example, in the theory of Fourier series, the projection \( P \) is associated with the \textbf{Dirichlet kernel}.

6. Let \( B_R = \{ x \in \mathbb{R}^3, |x|^2 < R^2 \} \) be the ball of radius \( R \) in \( \mathbb{R}^3 \) centered at the origin. Let \( a_0, a_1, a_2, a_3 \) be real numbers. Show that the solutions of the problem

\[
\begin{align*}
\Delta u &= a_1 x_1 + a_2 x_2 + a_3 x_3 + a_0 & \text{in } B_R, \\
u &= 0 & \text{at } \partial B_R, 
\end{align*} \tag{5}
\]

is a cubic polynomial in the variables \( x_1, x_2, x_3 \).

Hint: Search the solution as \( u(x) = (R^2 - x_1^2 - x_2^2 - x_3^2)(b_1 x_1 + b_2 x_2 + b_3 x_3 + b_0) \). You can also try to show a more general fact: if \( f \) is a polynomial of degree at most \( m \) in \( x_1, x_2, x_3 \), then the solution of \( \Delta u = f \) in \( B_R \) and \( u|_{\partial B_R} = 0 \) is of the form \( (R^2 - x_1^2 - x_2^2 - x_3^2) P(x_1, x_2, x_3) \), where \( P(x_1, x_2, x_3) \) is a polynomial of degree at most \( m \).