

Minimal initial data for potential Navier-Stokes singularities

W. Rusin* V. Šverák†

Abstract

Assuming some initial data $u_0 \in \dot{H}^{1/2}(R^3)$ lead to a singularity for the 3d Navier-Stokes equations, we show that there are also initial data with the minimal $\dot{H}^{1/2}$ -norm which will produce a singularity.

1 Introduction

We consider the Cauchy problem for the Navier-Stokes equations in $R^3 \times (0, \infty)$

$$\begin{aligned} u_t + u \nabla u + \nabla p - \Delta u &= 0 \\ \operatorname{div} u &= 0 \end{aligned} \quad \text{in } R^3 \times (0, \infty) \quad (1.1)$$

$$u(\cdot, 0) = u_0 \quad \text{in } R^3 \quad (1.2)$$

In this paper we will be interested in the case when the initial condition u_0 belongs to the space $\dot{H}^{1/2}(R^3)$. The $\dot{H}^{1/2}$ -norm is invariant under the natural scaling of the initial data $u_0(x) \rightarrow \lambda u_0(\lambda x)$, and the Cauchy problem is known to be globally well-posed for sufficiently small $u_0 \in \dot{H}^{1/2}$, and locally well-posed for any $u_0 \in \dot{H}^{1/2}$, as proved by Fujita and Kato [9]. These statements have to be made more precise by specifying the exact notion of the solution. The solutions constructed by Kato are usually called the mild solutions. See Section 3 for details. For $u_0 \in \dot{H}^{1/2}$ we denote by $T_{\max}(u_0)$ the maximal time of existence of the mild solution starting at u_0 .

*University of Minnesota. Supported in part by the Graduate School Dissertation Fellowship

†University of Minnesota. Supported in part by NSF Grant DMS-0800908

Let $\mathcal{B}_\rho = \{u_0 \in \dot{H}^{1/2}, \|u_0\|_{\dot{H}^{1/2}} < \rho\}$, and let ρ_{\max} be the supremum of all $\rho > 0$ such that the Cauchy problem 1.1–1.2 is globally well-posed for $u_0 \in \mathcal{B}_\rho$.

It is not known if ρ_{\max} is finite or infinite. Here we will be interested in the hypothetical situation when ρ_{\max} is finite. In principle ρ_{\max} could be finite for various reasons, which depend on the exact notion of the solution. However, one can show that with the natural definition of the mild solution, the only reason ρ_{\max} could be finite is the appearance of finite-time singularities in the solution u for some initial data u_0 .¹ We will consider the following question, motivated by a discussion of one of the authors with Isabelle Gallagher:

(Q) *If ρ_{\max} is finite, does there exist an initial datum $u_0 \in \dot{H}^{1/2}$ with $\|u_0\|_{\dot{H}^{1/2}} = \rho_{\max}$, such that the solution u of the Cauchy problem 1.1–1.2 develops a singularity in finite time?*

We will show that the answer to the question is affirmative, see Corollary 4.3.

The initial data u_0 with $\|u_0\|_{\dot{H}^{1/2}} = \rho_{\max}$ leading to a singularity will be called *$\dot{H}^{1/2}$ -minimal singularity-generating data*. We will show that, if singularities exist, the set of the $\dot{H}^{1/2}$ -minimal singularity-generating data is in fact a (nonempty) subset of $\dot{H}^{1/2}$ which is compact, modulo the action of the scalings $u_0(x) \rightarrow \lambda u_0(\lambda x)$ and translations $u_0(x) \rightarrow u_0(x - x_0)$.

Corollary 4.3 is an easy consequence of Theorem 4.2, and Lemma 2.1, which are in some sense the main new observation of this paper. Theorem 4.2 says, roughly speaking, that the solutions of the Cauchy problem are stable with respect to the weak convergence of the initial data.² This question was studied by I. Gallagher in [7] and Theorem 4.2 can be thought of as a continuation of those studies.

Lemma 2.1 says, roughly speaking, that singularities are stable under weak convergence of the solution in the energy norm.

Our results can also be used to show that the absence of singularities in all (reasonable) solutions is equivalent to certain a-priori estimates. Such statements were already proved in [7] and [22], and we give another illustration

¹The proof of the statement uses some special properties of the system 1.1–1.2, and can fail for other equations with similar non-linearities covered by the same perturbation theory, such as the complex viscous Burgers equation. In particular, the energy inequality plays an important role in the proof.

²There are several definitions of solutions and therefore one has to formulate the result with some care - see Section 4 for details.

of this principle.

Throughout this paper our main space for the initial data is the space $\dot{H}^{1/2}$, which is the unique \dot{H}^s space invariant under the natural scaling of the equation. It is natural to ask if our results are true for other scale-invariant spaces, such as L^3 , the Morrey space M with the norm with the norm $\|u\|_M^2 = \sup_{x,r} r^{-1} \int_{B_{x,r}} |u|^2$ studied in [23], or some other suitable spaces covered by [14]. We plan to address these questions in the future.

In the case of critical dispersive equation, the notion of minimal blow-up solutions (with a definition quite different from ours) and related profile decomposition has played an important role in the recent remarkable advances, see for example [2, 6, 12, 1]. These techniques have been recently also applied to the Navier-Stokes regularity in critical spaces, see [11].

The situation considered here is different, in that we focus only on the initial data, since we do not have bounds in critical norms for general solutions.

2 Suitable weak solutions

We first define *suitable weak solutions* of the Navier-Stokes equations, as introduced by [4]. See also [16] and [20]. This is a local notion. Let \mathcal{O} be an open subset of the space-time $R^3 \times R$ and let $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$, $p = p(x, t)$ be functions in \mathcal{O} such that

- u belongs locally to the *energy space* $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$,
- p belongs locally to the space $L_t^{3/2} L_x^{3/2}$,
- the equations $\operatorname{div} u = 0$ and $u_t + \operatorname{div}(u \otimes u) + \nabla p - \Delta u = 0$ are satisfied on \mathcal{O} in the sense of distributions , and
- the local energy inequality

$$2 \iint |\nabla u|^2 \phi \, dx \, dt \leq \iint [|u|^2 (\phi_t + \Delta u) + (|u|^2 + 2p) u \nabla \phi] \, dx \, dt \quad (2.1)$$

is satisfied for every non-negative smooth test function $\phi = \phi(x, t)$ compactly supported in \mathcal{O} .

In what follows we will use standard notation for euclidean balls centered at $x_0 \in R^n$ and parabolic balls $Q_{z_0,r}$ centered at $z_0 = (x_0, t_0) \in R^n \times R$:

$$\begin{aligned} B_{x_0,r} &= \{x \in R^n; |x - x_0| < r\} \\ Q_{z_0,r} &= B_{x_0,r} \times (t_0 - r^2, t_0] \end{aligned}$$

Given a suitable weak solution (u, p) , a point $z_0 = (x_0, t_0) \in \mathcal{O}$ is called a *regular point* of (u, p) if u is Hölder continuous in a neighborhood of z_0 . A *singular point* $z_0 \in \mathcal{O}$ of (u, p) is any point which is not regular. We will use the following two propositions, the various versions of which can be found in [4, 16, 20, 15]. The version below contains some quantitative estimates which are often not explicitly stated in the literature, although they are implicit in the proofs. A sketch of the proof of the spatial derivatives estimates can be found for example in [19].

Proposition 2.1. (ε -regularity criterion) *There exists $\varepsilon_0 > 0$ such that the following statement is true:*

If (u, p) is a suitable weak solution in \mathcal{O} , such that

$$\frac{1}{r^2} \iint_{Q_{z_0,r}} (|u|^3 + |p|^{3/2}) dx dt < \varepsilon_0 , \quad (2.2)$$

for some $Q_{z_0,r}$ compactly contained in \mathcal{O} , then all points in $Q_{z_0,r/2}$ are regular points of (u, p) . Moreover, in $Q_{z_0,r/2}$ one has

$$|\nabla^k u| \leq C_k r^{-1-k} \quad k = 0, 1, \dots \quad \text{and} \quad (2.3)$$

$$|u(x, t) - u(x, t')| \leq C' |t - t'|^{1/3} \quad (2.4)$$

Remark: The regularity in t cannot be improved, due to solutions of the form $u(x, t) = \nabla h(x, t)$ with h harmonic in x and having arbitrary dependence on t . The Hölder exponent in t for these solutions is dictated by the assumptions on the integrability of the pressure $p = -|\nabla h|^2/2 - h_t$, and under the assumptions of the lemma the Hölder exponent $1/3$ is optimal.

Proposition 2.2. (Compactness) *Let (u^k, p^k) , $k = 1, 2, \dots$ be a sequence of suitable weak solutions such that u^k are uniformly bounded in the energy space $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$ on compact subsets of \mathcal{O} and p^k are uniformly bounded in $L_t^{3/2} L_x^{3/2}$ on compact subsets of \mathcal{O} . Then the sequence u^k is compact in $L_t^3 L_x^3$ on compact subsets of \mathcal{O} . Moreover, if $u^k \rightarrow u$ in $L_t^3 L_x^3$ on compact subsets of \mathcal{O} and $p^k \rightarrow p$ in $L_t^{3/2} L_x^{3/2}$ on compact subsets of \mathcal{O} , then (u, p) is again a suitable weak solution.*

The two previous propositions imply the following lemma, which will be important for the proof of our main results.

Lemma 2.1. (Stability of singularities) *In the situation of Proposition 2.2, assume that $z^k \in \mathcal{O}$ are singular points of (u^k, p^k) , $k = 1, 2, \dots$, and that $z^k \rightarrow z_0 \in \mathcal{O}$. Then z_0 is a singular point of (u, p) .*

Proof. If the regularity criterion in Proposition 2.1 did not contain the pressure p , the statement of the Lemma would be immediate: indeed, if z_0 is a regular point of u , then $r^{-2} \iint_{Q_{z_0,r}} |u|^3 dx dt = O(r^3)$ as $r \rightarrow 0_+$. Choosing a sufficiently small r , one sees that $r^{-2} \iint_{Q_{z_0,r}} |u^k|^3 dx dt$ is small for large k by the strong convergence of u^k in $L_{t,x}^3$. However, such argument cannot be applied to the pressure term, since the sequence p^k may not have a subsequence which is compact in $L_{t,x}^{3/2}$. It is well known how to deal with this difficulty: the trick can be found in one form or another in the proofs of partial regularity [4, 16, 20, 15]. The pressure p^k solves the equation

$$-\Delta p^k = \partial_i \partial_j (u_i^k u_j^k) \quad (2.5)$$

Recall that the term $u_i^k u_j^k$ is compact in $L_{t,x}^{3/2}$ on compact subsets of \mathcal{O} . Therefore we can invert the Laplacian in 2.5 using a suitable boundary condition, (or just taking the Riesz transforms $\tilde{p}^k = R_i R_j (u_i^k u_j^k \chi_{B_{x_0,r}})$) and decompose p^k as

$$p^k = \tilde{p}^k + h^k \quad (2.6)$$

with \tilde{p}^k compact in $L_{t,x}^{3/2}(Q_{z_0,r})$ (by Calderon-Zygmund estimates) and h^k bounded in $L_{t,x}^{3/2}(Q_{z_0,r})$ and harmonic in x in $Q_{z_0,r}$. The term with \tilde{p}^k can be dealt with in the same way as the term with u^k . The term h^k is handled by using classical estimates for harmonic functions: let $\gamma \geq 1$ and let $h \in L_x^\gamma(B_{x_0,r})$ be harmonic in $B_{x_0,r}$. We denote $(h)_{r'} = |B_{x_0,r'}|^{-1} \int_{B_{x_0,r'}} h$. For $r' \leq r/2$ and $x' \in B_{x_0,r'}$ we have

$$|h(x') - (h)_{r'}|^\gamma \leq C_\gamma \left(\frac{r'}{r} \right)^\gamma r^{-3} \int_{B_{x_0,r}} |h|^\gamma dx \quad (2.7)$$

We recall that we can change the pressure by any function depending on t only. Therefore we can use 2.7 with $h = h^k$, and integrating over $Q_{z_0,r'}$, we get the required smallness of the term $(r')^{-2} \iint_{Q_{z_0,r'}} |h^k - (h^k(\cdot, t))_{r'}|^{3/2} dx dt$. \square

In fact, the above proof together with the estimates in Proposition 2.1 give the following version of Lemma 2.1.

Lemma 2.2. *Under the assumptions of Proposition 2.2, let K be a compact subset of \mathcal{O} . If each point of K is a regular point of u , then, for sufficiently large k , each point of K is also a regular point of u^k , and on the set K the functions u^k converge to u , together with all spatial derivatives.*

3 Mild solutions

In this section we review the results we need about the so-called “mild solutions” of the problem 1.1– 1.2. This approach was introduced by Fujita and Kato, [9], see also [10], although the terminology was introduced later. Let us first recall basic facts about the linear Stokes problem

$$\begin{aligned} u_t + \nabla p - \Delta u &= \frac{\partial}{\partial x_k} f_k \\ \operatorname{div} u &= 0 \end{aligned} \quad \left. \right\} \quad \text{in } R^n \times (0, \infty) \quad (3.1)$$

$$u(\cdot, 0) = u_0 \quad \text{in } R^n \quad (3.2)$$

Here $f_k = (f_{1k}, \dots, f_{nk})$ for $k = 1, \dots, n$. Let $S(t)$ be the solution operator of the heat equation and let P be the Helmholtz projection of vector fields onto the divergence-free vector fields. By definition, a mild solution of the linear problem above is given by the representation formula

$$u(t) = S(t)u_0 + \int_0^t S(t-s)P\nabla \cdot f(s) ds \quad (3.3)$$

A mild solution of the Cauchy problem 1.1– 1.2 is the mild solution of the linear problem above with $f_{ij} = -u_i u_j$. We will denote the “heat extension” $S(t)u_0$ of the initial datum u_0 by $U = U(x, t)$. The term $\int_0^t S(t-s)P\nabla \cdot f(s) ds$ with $f_{ij} = -u_i v_j$ will be denoted by $B(u, v)$. In this notation, a mild solution of the Cauchy problem 1.1– 1.2 in $R^3 \times (0, T)$ is defined as a solution of the integral equation

$$u = U + B(u, u) \quad (3.4)$$

in a suitably defined space of functions X on $R^3 \times (0, T)$. In this approach, a key property of X is the continuity of the bilinear form $(u, v) \rightarrow B(u, v)$ as a map from $X \times X$ to X . This is equivalent to

$$\|B(u, v)\|_X \leq c\|u\|_X\|v\|_X \quad (3.5)$$

For initial datum $u_0 \in \dot{H}^{1/2}$ there are many possible choices of X . A good choice is for example $X = L_t^4 \dot{H}_x^1$. In this case the proof of 3.5 is particularly simple: using the inequality $\|fg\|_{\dot{H}^{1/2}(R^3)} \leq c\|f\|_{\dot{H}^1(R^3)}\|g\|_{\dot{H}^1(R^3)}$ we see that for $u, v \in X$ we have $uv \in L_t^2 \dot{H}_x^{1/2}$. Recalling the energy inequality for the linear system 3.1 ,

$$\| |\nabla|^s u \|_{L_t^\infty L_x^2}^2 + \| |\nabla|^s u \|_{L_t^2 \dot{H}_x^1}^2 \leq \| |\nabla|^s u_0 \|_{L_x^2}^2 + \| |\nabla|^s f \|_{L_t^2 L_x^2}^2 \quad (3.6)$$

one easily gets 3.5. Also, the energy inequality implies that $u_0 \in \dot{H}^{1/2}$ gives $U \in X$, with $\|U\|_X \leq \|u_0\|_{\dot{H}^{1/2}}$.

In fact, the above proof gives that B maps $L_t^4 \dot{H}_x^1 \times L_t^4 \dot{H}_x^1$ into $C_t \dot{H}_x^{1/2} \cap L_t^2 \dot{H}_x^{3/2}$ (where the first space denotes the space of continuous functions of t with values in $\dot{H}^{1/2}$), which shows that equation 3.4 can be treated as an ODE in t . In particular, one always has local-in-time existence of the solution u , and one can define the maximal time of existence $T_{\max}(u_0)$ on which the solution of 3.4 exists.

If $T_{\max}(u_0)$ is finite, one has

$$\lim_{T \rightarrow T_{\max}(u_0)} \|u\|_{L_t^4 \dot{H}_x^1(R^3 \times (0, T))} = \infty. \quad (3.7)$$

We note that for sufficiently small $\|u_0\|_{\dot{H}^{1/2}}$ we have $T_{\max}(u_0) = +\infty$. This justifies the definition of ρ_{\max} from the Introduction.

The mild solutions have the same regularity as U since, roughly speaking, for short times they are just a perturbation of U , and this can be iterated forward to the time interval where the solution exists.³ In particular, the mild solutions are smooth in $R^3 \times (0, T_{\max}(u_0))$.

One obvious reason for $T_{\max}(u_0)$ to be finite would be the development of a singularity in the solution u at time $T_{\max}(u_0)$. A-priori it is not clear that this is the only reason. One could also imagine a scenario where the $L_t^4 \dot{H}_x^1$ norm of the solution would blow up even though the solution would remain smooth on each compact subset. However, this scenario can be ruled out. The only reason for the blow-up of the $L_t^4 \dot{H}_x^1$ norm of the mild solutions u with $u_0 \in \dot{H}^{1/2}$ are the possible finite-time singularities. This will be justified in the next section.

³One has to be cautious with this statement if “regularity” also means decay properties of u as $x \rightarrow \infty$. Due to the non-local effect of the constraint $\operatorname{div} u = 0$, the solutions u can have slower decay at infinity than U . See for example [3].

4 Leray's solutions

In his pioneering work [18] Leray proved the existence of the weak solutions to the Cauchy problem 1.1– 1.2. A key ingredient in his approach is the energy inequality

$$\int_{R^3} |u(x, t)|^2 dx + 2 \int_0^t \int_{R^3} |\nabla u(x, s)|^2 dx ds \leq \int_{R^3} |u_0(x)|^2 dx \quad (4.1)$$

This inequality is the only known a-priori bound for general solutions. At the first glance it would seem that for its application it is crucial that $u_0 \in L^2(R^3)$, which would rule out using Leray's techniques in the situation of the preceding section, where the basic assumption is $u_0 \in \dot{H}^{1/2}$, which is not a subset of L^2 .

However, Lemarié-Rieusset [17] found a generalization of 4.1 to the situation when the energy is only (uniformly) locally finite, and this makes it possible to extend the theory of Leray's weak solutions to a much more general setting. See also [13]. In this paper we will not need the full version of Lemarié-Rieusset's local theory, but we will need a version of his inequality for local energy, see Lemma 4.1.

In our setting with $u_0 \in \dot{H}^{1/2}$ one can use the following trick by C. Calderon [5] to construct the weak solutions in a simple way. We can write $u_0 = a_0 + v_0$ with a_0 smooth and small in $\dot{H}^{1/2}$, and v_0 in L^2 . (For example, a_0 can be defined in terms of the Fourier transform as $\hat{a}_0(\xi) = \hat{u}_0(\xi)\varphi(\xi)$, where φ is a suitable smooth function equal to 1 in a small neighborhood of 0.) Since a_0 is small, the Cauchy problem 1.1– 1.2 has a global solution a which is small in $L_t^4 \dot{H}_x^1$. We now seek solutions u in the form $u = a + v$, where v is a new unknown function satisfying the equation

$$v_t + a\nabla v + v\nabla a + v\nabla v + \nabla q - \Delta v = 0. \quad (4.2)$$

The energy identity for this equation is

$$\int_{R^3} |v(x, t)|^2 dx + 2 \int_0^t \int_{R^3} |\nabla v(x, s)|^2 dx ds = \int_{R^3} |v_0(x)|^2 dx + 2 \int_0^t \int_{R^3} (a\nabla v)v. \quad (4.3)$$

The last integral on the right-hand side can be estimated by

$$c\|a\|_{L_t^4 \dot{H}_x^1} \|v\|_{L_t^\infty L_x^2}^{1/2} \|\nabla v\|_{L_t^2 L_x^2}^{3/4}$$

and we see that we have a good energy estimate for v when $\|a\|_{L_t^4 \dot{H}_x^1}$ is sufficiently small.

Leray's construction of weak solutions can therefore be applied to equation 4.2 for v . This way we can construct a global weak solution $u = a + v$ to the Cauchy problem with $u_0 \in \dot{H}^{1/2}$. The pressure can be recovered from the equation $-\Delta p = \partial_i \partial_j u_i u_j$. Moreover, one can do the construction in such a way that (u, p) is a suitable weak solution in $R^3 \times (0, \infty)$ and $u(t) \rightarrow u_0$ in L^2 on every compact subset of R^3 . The weak solution u with these properties will be called *the Leray solution*. The relation of the Leray solution and the mild solution introduced in the previous section is clarified by the following "weak-strong uniqueness" theorem. In the case $u_0 \in L^2 \cap \dot{H}^1$ the theorem was proved by Leray. Leray's result was generalized in various directions, see for example [21, 24, 17]. We will use the following version which is a special case of Theorem 33.2, p. 354 from Lemarié-Rieusset's book [17].

Theorem 4.1. *Let u be a Leray solution of the initial value problem 1.1–1.2 with $u_0 \in \dot{H}^{1/2}$. Let $T_{\max}(u_0)$ be the maximal time of existence of the mild solution with of 1.1–1.2 with the same initial value u_0 . Then the mild solution coincides with u in $R^3 \times [0, T_{\max}(u_0))$.*

The problem of uniqueness of u after $T_{\max}(u_0)$ is open. At the time of this writing we cannot rule out that $T_{\max}(u_0)$ is finite and that the Leray solution is not unique after $T_{\max}(u_0)$. We will denote the set of all Leray solutions with initial data $u_0 \in \dot{H}^{1/2}$ by $\text{NS}(u_0)$.

Proposition 2.1 can be used to show that the only reason for $T_{\max}(u_0) < \infty$ can be a finite time singularity. We will now sketch a proof of this statement. Let us assume that $T = T_{\max}(u_0)$ is finite. Set $r = \sqrt{T/2}$. We consider the decomposition $u = a + v$ as above, where a is a solution generated by a_0 with small $\|a_0\|_{\dot{H}^{1/2}}$ (and hence a satisfies global estimates) and v satisfying the energy estimates. The key point is that these estimates do not deteriorate as we approach T . Using these estimates, together with corresponding estimates for the pressure, it is not hard to see that for sufficiently large $R > 0$, the assumptions of Proposition 2.1 are satisfied for our solution (u, p) and $Q_{z_0, r}$ with $z_0 = (x_0, T)$ and $|x_0| > R$. If u does not develop a singularity at time T in the ball B_R , it means that u and ∇u will be bounded in (t_1, T) for any $t_1 > 0$. We can now write the Navier-Stokes equation for u as

$$u_t - \Delta u + \nabla p = -\operatorname{div}(u \otimes u) . \quad (4.4)$$

Using the standard estimates for the small solution a , the energy estimates for v , together with the pointwise bound for u and ∇u , one can easily show that the term $u \otimes u = (a+v) \otimes (a+v)$ is bounded both in $L_t^2 L_x^2(R^3 \times (T/2, T))$ and $L_t^2 \dot{H}_x^1(R^3 \times (T/2, T))$, and therefore also in $L_t^2 \dot{H}_x^{1/2}(R^3 \times (T/2, T))$. Viewing 4.4 as a linear equation with the right-hand side $-\operatorname{div}(u \otimes u)$, we see by the energy estimate that $u \in L_t^4 \dot{H}_x^1(R^3 \times (0, T))$, which means that T was not the maximal time of existence of the mild solution, a contradiction.

The weak solution v of equation 4.2 always belongs to the energy space $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1$. As noticed already by Leray in [18], this implies that v is smooth and small for large times. In our set-up we can see this from the fact that $\|v(t)\|_{\dot{H}^{1/2}}^2 \leq \|v(t)\|_{L^2} \|\nabla v(t)\|_{L^2}$ and the expression on the right-hand side of this inequality clearly has to be small on a large set of times if v is in the energy space. Following the reasoning of Leray, ([18], p. 246), we notice that at a time t_0 when $\|v(t_0)\|_{\dot{H}^{1/2}}$ is small we can apply the theory of mild solutions and the weak-strong uniqueness results to see that after time t_0 the solution v coincides with a global mild solutions. Similar considerations have been used for example in [8].

We can summarize the above facts in the following statement:

Proposition 4.1. *Let u be a Leray solution of the Cauchy problem 1.1–1.2 with $u_0 \in \dot{H}^{1/2}$. Then for some compact set $K \subset R^3 \times (0, \infty)$ we have $\nabla u \in L_t^4 L_x^2(R^3 \times (0, \infty) \setminus K)$. In particular, u is regular at every point of $R^3 \times (0, \infty) \setminus K$.*

Proof. The proof of the first statement follows from the comments above. The second statement follows from the first and the standard regularity criteria, such as the Ladyzhenskaya-Prodi-Serrin criterion, or from Proposition 2.1. \square

The energy estimate for v which can be obtained from equation 4.2 depends on the decomposition of the initial data $u_0 = a_0 + v_0$. For our purposes in this paper we need an energy estimate which is “more uniform” (although more local). Fortunately, an estimate found by Lemarié-Rieusset in his work on weak solutions with locally finite energy provides exactly what we need. We will use the following notation: for $x_0 \in R^3$ and $r > 0$ we will denote by $\tilde{Q}_{x_0, r}$ the space-time cylinder $B_{x_0, r} \times (0, r^2)$. We will also use the notation $\|u\|_{\mathcal{E}(\tilde{Q}_{x_0, r})}$ to denote the energy norm defined by

$$\|u\|_{\mathcal{E}(\tilde{Q}_{x_0, r})}^2 = \|u\|_{L_t^\infty L_x^2(\tilde{Q}_{x_0, r})}^2 + \|\nabla u\|_{L_t^2 L_x^2(\tilde{Q}_{x_0, r})}^2. \quad (4.5)$$

Lemma 4.1. *Let $u_0 \in \dot{H}^{1/2}$ and let u be a Leray solution of the Cauchy problem 1.1–1.2 with initial condition u_0 . Then for each $r > 0$ and $x_0 \in R^3$*

$$\|u\|_{\mathcal{E}(\tilde{Q}_{x_0,r})}^2 \leq C(\|u_0\|_{\dot{H}^{1/2}}) r \quad (4.6)$$

and, for a suitable function $p_{x_0,r}(t)$ of t ,

$$\iint_{\tilde{Q}_{x_0,r}} |p - p_{x_0,r}(t)|^{3/2} dx dt \leq C(\|u_0\|_{\dot{H}^{1/2}}) r^2 \quad (4.7)$$

Proof. The first estimate can be easily derived from Proposition 32.1, p. 342 and its proof in [17], and the second estimate follows from Lemma 32.2, p. 343 in the same book. There are two crucial points in the proof of these estimates. One is that the energy flux in the localized energy estimate 2.1 is bounded by $\sim |u|^3$, if we count the pressure as $p \sim |u|^2$. The energy itself controls $\sim |u|^{10/3}$, and it is important for the proof that this is a higher power than the one in the energy flux. This is no longer the case in higher dimensions and therefore a possible generalization to higher dimensions would not be straightforward. Similar issue arises in the proof of partial regularity. The second point is that the non-local effects of the pressure are under control, so that the heuristics $p \sim |u|^2$ is valid, at least as far as the estimates are concerned. To see this, we notice that the kernel of the pressure equation

$$-\Delta p = \partial_i \partial_j (F_{ij}), \quad \text{with } F_{ij} = u_i u_j \quad (4.8)$$

is

$$G_{ij} = \partial_i \partial_j G, \quad \text{with } G(x) = \frac{1}{4\pi|x|}. \quad (4.9)$$

Therefore the kernel for expressing the gradient ∇p in terms of F_{ij} decays as $|x|^{-4}$ as $x \rightarrow \infty$, and is integrable near ∞ . This fast decay makes it possible to estimate the contributions to ∇p from far away. This would not be the case for p , and hence we have to work with ∇p , which is the reason for the appearance of the function $p_{x_0,r}$ in the estimate 4.7. This part of the argument would work in the higher dimensions as well. \square

We can now formulate the main new result of this section.

Theorem 4.2. *Let u_0^k be a bounded sequence of initial conditions in $\dot{H}^{1/2}$ converging weakly in $\dot{H}^{1/2}$ to u_0 . Let $u_k \in \text{NS}(u_0^k)$ be Leray solutions of the Cauchy problem with initial conditions u_0^k . Assume that u^k converge weakly to u in distributions. Then $u \in \text{NS}(u_0)$, i. e. u is a Leray solution of the Cauchy problem with initial condition u_0 .*

Proof. Using Lemma 4.1, Proposition 2.2 and Theorem 4.1, we see that it is enough to show that $u(t) \rightarrow u_0$ in L^2 on every compact subset of \mathbb{R}^3 . This can be seen as follows. We take a non-negative smooth test function $\phi(x, t)$ compactly supported in $\mathbb{R}^3 \times [0, \infty)$. Note that we are taking the interval $[0, \infty)$, which is closed at zero, and $\phi(x, 0)$ does not have to vanish everywhere. We write a version of the local energy inequality with such test functions in the following form.

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^3} [-|u^k|^2 \phi_t + 2|\nabla u^k|^2 \phi] \, dx dt \leq \\ & \quad \int_{\mathbb{R}^3} |u_0^k|^2 \phi(x, 0) \, dx + \int_0^\infty \int_{\mathbb{R}^3} [|u^k|^2 \Delta \phi + (|u^k|^2 + 2p^k) u^k \nabla \phi] \, dx dt \end{aligned} \quad (4.10)$$

Since for every compactly supported smooth test function ψ the sequence $u_0^k \psi$ is compact in L^2 , we see that in the limit $k \rightarrow \infty$ the inequality 4.10 will be preserved. Hence

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^3} [-|u|^2 \phi_t + 2|\nabla u|^2 \phi] \, dx dt \leq \\ & \quad \int_{\mathbb{R}^3} |u_0|^2 \phi(x, 0) \, dx + \int_0^\infty \int_{\mathbb{R}^3} [|u|^2 \Delta \phi + (|u|^2 + 2p) u \nabla \phi] \, dx dt, \end{aligned} \quad (4.11)$$

where p is a suitable pressure corresponding to the solution u . The last inequality implies the required local L^2 -continuity property at time $t = 0$ for the solution u . The key points, well-known in the theory of weak solutions and going back to Leray's paper [18] are that

- (a) the equation implies that $u(t) \rightarrow u_0$ weakly in L^2 on compact subsets as $t \rightarrow 0$ and
- (b) inequality 4.11 implies that for every compactly supported smooth test function ψ one has $\limsup_{t \rightarrow 0+} \|u(t)\psi\| \leq \|u_0\psi\|$.

□

Corollary 4.1. *Let u_0^k, u_0, u^k be as in Theorem 4.2. Let $(0, T_{\max}(u_0))$ be the maximal interval of existence of the mild solution u starting at u_0 . Then for any compact set $K \subset \mathbb{R}^3 \times (0, T_{\max}(u_0))$ and any $k \geq k_0 = k_0(K)$ the solutions u^k are regular at all points of K and converge uniformly to u in K , together with all spatial derivatives.*

Proof. Apply the theorem, together with Lemma 4.1, Proposition 2.2 and Lemma 2.2. □

Corollary 4.2. Let u_0^k , u_0 , u^k be as in the theorem. Assume that $T_{\max}(u_0^k) = T < +\infty$ for each k and that the singular points z_k of u at $t = T$ (which exist by Proposition 4.1) stay in a compact subset of $R^3 \times \{T\}$. Then $T_{\max}(u_0) \leq T$.

Proof. Apply the theorem, together with Lemma 4.1, Proposition 2.2 and Lemma 2.1. \square

Let us recall that

$$\rho_{\max} = \sup\{\rho; T_{\max}(u_0) = +\infty \text{ for every } u_0 \in \dot{H}^{1/2} \text{ with } \|u_0\|_{\dot{H}^{1/2}} < \rho\} \quad (4.12)$$

Let us also define

$$\mathcal{M} = \{u_0 \in \dot{H}^{1/2}; T_{\max}(u_0) < \infty, \|u_0\|_{\dot{H}^{1/2}} = \rho_{\max}\} \quad (4.13)$$

Corollary 4.3. The set \mathcal{M} is non-empty. Moreover, \mathcal{M} is compact modulo scalings and translations, i. e. if $u_0^k \in \mathcal{M}$ is a sequence in \mathcal{M} , then there exist $\lambda_k > 0$ and $x_0^k \in R^3$ such the sequence $v^k \in \dot{H}^{1/2}$ defined by $v^k(x) = \lambda_k u_0^k(\lambda_k x - x_0^k)$ is compact in $\dot{H}^{1/2}$.

Proof. Let $u_0^k \in \dot{H}^{1/2}$ be a sequence of initial data with $T_{\max}(u_0^k)$ finite and $\|u_0^k\|_{\dot{H}^{1/2}} \rightarrow \rho_{\max}$. Find $\lambda_k > 0$ and x_0^k so that the functions given by $v^k(x) = \lambda_k u_0^k(\lambda_k x - x_0^k)$ develop their first singularity at time $t = 1$ and that $(x, t) = (0, 1)$ is a singular point of v^k . We can assume that the functions $v_0^k(x) = v^k(x, 0)$ converge weakly in $\dot{H}^{1/2}$ to $v_0 \in \dot{H}^{1/2}$. By Corollary 4.2 we know that $T_{\max}(v_0) \leq 1$, and by definition of ρ_{\max} this means that $\|v_0\|_{\dot{H}^{1/2}} = \rho_{\max}$. This shows that \mathcal{M} is non-empty. We also see that $\|v_0^k\| \rightarrow \|v_0\|$ and hence $v_0^k \rightarrow v_0$ strongly. \square

The following corollary can be thought of as a variant of results in [7] and [22]. See also Theorem 33.3, p. 359 in [17] for a related statement about “individual solutions”.

Corollary 4.4. Assume that every solution of the Cauchy problem 1.1–1.2 with $u_0 \in \dot{H}^{1/2}$ is regular, i. e. $T_{\max}(u_0) = +\infty$ for each $u_0 \in \dot{H}^{1/2}$. Then, for $l = 0, 1, 2, \dots$ there exist functions $F_l: [0, \infty) \rightarrow [0, \infty)$ such that

$$t^{(l+1)/2} \sup_x |\nabla^l u(x, t)| \leq F_l(\|u_0\|_{\dot{H}^{1/2}}) \quad (4.14)$$

Proof. Let us prove the case $l = 0$, the proof for the derivatives being the same. By scaling invariance, it is enough to prove the statement for $t = 1$. If the statement fails we can assume by the translational invariance that there exists a sequence of initial data u_0^k bounded in $\dot{H}^{1/2}$ such that for the corresponding solutions u^k one has $|u^k(0, 1)| \geq k$. Let u_0 be a weak limit of u_0^k . By our assumption, the solution u corresponding to u_0 is regular at $(x, t) = (0, 1)$ and by Theorem 4.2 and Lemma 2.2 we get a contradiction. \square

References

- [1] Bahouri, H., Gerard, P., High frequency approximation of solutions to critical nonlinear wave equations. Amer. J. Math. 121 (1999), no. 1, 131–175.
- [2] Bourgain, J., Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. J. Amer. Math. Soc. 12 (1999), no. 1, 145–171.
- [3] Brandoles, L., Localisation, oscillation et comportement asymptotique pour les équations de Navier-Stokes, These de doctorat, ENS Cachan (2001).
- [4] Caffarelli, L., Kohn, R.-V., Nirenberg, L., Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math., Vol. XXXV (1982), pp. 771–831.
- [5] Calderon, C. P., Existence of weak solutions for the Navier-Stokes equations with initial data in L^p . Trans. Amer. Math. Soc. 318 (1990), no. 1, 179–200.
- [6] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T., Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in \mathbb{R}^3 . Ann. of Math. (2) 167 (2008), no. 3, 767–865.
- [7] Gallagher, I., Profile decomposition for solutions of the Navier-Stokes equations. Bull. Soc. Math. France 129 (2001), no. 2, 285–316.
- [8] Gallagher, I., Iftimie, D., Planchon, F., Asymptotics and stability for global solutions to the Navier-Stokes equations. Ann. Inst. Fourier (Grenoble) 53 (2003), no. 5, 1387–1424.

- [9] Kato, T., Fujita, H., On the nonstationary Navier-Stokes system. *Rend. Sem. Mat. Univ. Padova* 32 1962 243–260.
- [10] Kato, T. Strong L^p -solutions of the Navier-Stokes equation in \mathbb{R}^m , with applications to weak solutions, *Math. Z.* 187 (1984), no. 4, 471–480.
- [11] Kenig, C., Koch, G., An alternative approach to regularity for the Navier-Stokes equations in a critical space, preprint arXiv:0908.3349.
- [12] Kenig, C., Merle, F., Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schroedinger equation in the radial case. *Invent. Math.* 166 (2006), no. 3, 645–675.
- [13] Kikuchi, N., Seregin, G., Weak solutions to the Cauchy problem for the Navier-Stokes equations satisfying the local energy inequality. *Nonlinear equations and spectral theory*, 141–164, Amer. Math. Soc. Transl. Ser. 2, 220, Amer. Math. Soc., Providence, RI, 2007.
- [14] Koch, H., Tataru, D. Well-posedness for the Navier-Stokes equations, *Adv. Math.* 157 (2001), no. 1, 22–35.
- [15] Ladyzhenskaya, O. A., Seregin, G. A., On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. *J. Math. Fluid Mech.* 1 (1999), no. 4, 356–387.
- [16] Lin, F-H., A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.* 51 (1998), no. 3, 241–257.
- [17] Lemarié-Rieusset, P. G., Recent developments in the Navier-Stokes problem. Chapman & Hall/CRC Research Notes in Mathematics, 431. Chapman & Hall/CRC, Boca Raton, FL, 2002.
- [18] Leray, J., Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.* 63(1934), pp. 193–248.
- [19] Nečas, J.; Růžička, M.; Šverák, V. On Leray's self-similar solutions of the Navier-Stokes equations. *Acta Math.* 176 (1996), no. 2, 283–294.
- [20] Scheffer, V., Partial regularity of solutions to the Navier-Stokes equations. *Pacific J. Math.* 66 (1976), no. 2, 535–552.

- [21] Serrin, J., The initial value problem for the Navier-Stokes equations. 1963 Nonlinear Problems (Proc. Sympos., Madison, Wis. pp. 69–98 Univ. of Wisconsin Press, Madison, Wis.
- [22] Tao, T., A quantitative formulation of the global regularity problem for the periodic Navier-Stokes equation. *Dyn. Partial Differ. Equ.* 4 (2007), no. 4, 293–302.
- [23] Taylor, M. E., Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations. *Comm. Partial Differential Equations* 17 (1992), no. 9-10, 1407–1456.
- [24] von Wahl, W., The equations of Navier-Stokes and abstract parabolic equations. Aspects of Mathematics, E8. Friedr. Vieweg & Sohn, Braunschweig, 1985.