

# On Type I singularities of the local axi-symmetric solutions of the Navier-Stokes equations

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**Abstract** Local regularity of axially symmetric solutions to the Navier-Stokes equations is studied. It is shown that under certain natural assumptions there are no singularities of Type I.

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**Key Words:** Navier-Stokes equations, regularity, axial symmetry.

## 1 Introduction

In this paper we will consider local regularity properties of axi-symmetric solutions of the 3D Navier-Stokes equations

$$\begin{aligned} \partial_t v + v \cdot \nabla v + \nabla q - \Delta v &= 0 \\ \operatorname{div} v &= 0. \end{aligned} \tag{1.1}$$

Most of the known regularity theory for these equations (and, in fact, for many other equations) is based on optimal estimates for the linear part and

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on treating the nonlinearity as a perturbation which is (locally) small in a suitable sense. An important role in formulating suitable smallness conditions is played by certain (local) scale-invariant quantities. These are the quantities which are invariant under the scaling symmetry  $v(x, t), q(x, t) \rightarrow \lambda v(\lambda x, \lambda^2 t), \lambda^2 q(\lambda x, \lambda^2 t)$ . The reason why the regularity criteria should be formulated in terms of the scale-invariant quantities is simple: The class of regular solutions is invariant under the scaling and therefore sufficient conditions for membership in this class should ideally be also invariant under the scaling, or at least they should scale in the correct way, in that the quantity controlling regularity should not decrease if we scale the solution with  $\lambda \gg 1$ .

To write down examples of the scale-invariant quantities we recall the following standard notation. The points of the space-time  $\mathbb{R}^n \times \mathbb{R}$  will be denoted by  $z = (x, t)$ . For  $x_0 \in \mathbb{R}^n$  we denote by  $B(x_0, R)$  the ball  $\{x : |x - x_0| < R\}$  and for  $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$  we denote by  $Q(z_0, R)$  the parabolic ball  $B(x_0, R) \times ]t_0 - R^2, t_0[$ . Here are some examples of the scale-invariant quantities for  $n = 3$ :

$$\int_{Q(z_0, R)} |v|^5 dz , \tag{1.2}$$

$$\text{ess sup}_{t \in ]t_0 - R^2, t_0[} \int_{B(x_0, R)} |v(x, t)|^3 dx , \tag{1.3}$$

$$R^{-2} \int_{Q(z_0, R)} |v|^3 dz , \tag{1.4}$$

$$R^{-3} \int_{Q(z_0, R)} |v|^2 dz , \tag{1.5}$$

$$R^{-1} \int_{Q(z_0, R)} |\nabla v|^2 dz , \tag{1.6}$$

$$\text{ess sup}_{(x, t) \in Q(z_0, R)} \sqrt{t_0 - t} |v(x, t)| , \tag{1.7}$$

$$\text{ess sup}_{(x, t) \in Q(z_0, R)} |x - x_0| |v(x, t)| . \tag{1.8}$$

A typical local regularity result says that, under some natural technical

assumptions <sup>1</sup>, a point  $z_0$  is a regular point of the solution  $v$  if a suitable scale invariant quantity  $X(z_0, R; v)$  of the type in the examples above is sufficiently small for all  $R \in ]0, R_0[$ . In fact,  $X$  can be any of the quantities above with the exception of (1.5), in which case the validity of the corresponding result is open. <sup>2</sup>

At the time of this writing, there is no known scale-invariant quantity for which an a-priori estimate would be known for general 3D solutions. In fact, all the known estimates can be traced back to the energy estimate, which gives bounds in quantities such as  $\int_{B(x_0, R)} |v(x, t)|^2 dx$  or

$\int_{Q(z_0, R)} |\nabla v|^2 dz$ , which do not have the scaling needed for the existing local regularity theory. This is often quoted as the main stumbling block in our understanding of the Navier-Stokes regularity. This statement is probably correct, at least as a first approximation. However, even if we *assume* that scale-invariant estimates of natural quantities are available, in many cases we are still unable to prove regularity, unless an additional smallness condition is imposed. For the quantities of the type  $X(z_0, R; v)$  listed above one can show that  $X(z_0, R; v) \leq C$  for some  $C > 0$  (not necessarily small) implies regularity for (1.2) and (1.3), but in the remaining cases the known theory requires an additional smallness condition (and, as remarked above, the situation with (1.5) is even worse). Moreover, even the proofs of the cases (1.2) and (1.3) rest on the fact that the assumptions imply that a certain quantity becomes small.<sup>3</sup>

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<sup>1</sup>Such assumption must include some control of pressure, such as  $q \in L_{\frac{3}{2}}$ . Fortunately, such control is available from energy estimates in most situations.

<sup>2</sup>The reason for the difficulties with (1.5) is that the space-time  $L_2$  norm of  $v$  is not sufficiently strong to control the energy flux (unless one can come up with some surprising new property of the equations). Roughly speaking, the energy flux is controlled by the  $L_3$  norm of  $v$ . Since the energy estimate gives the control of the  $L_{\frac{10}{3}}$  norm of  $v$ , there is some gain in regularity and it is natural to try to bootstrap it and try control the energy flux by some  $L_r$  norm with  $r < 3$ . This does work, but the borderline exponent  $r$  for this argument is  $r = 5/2$ , still quite far from  $r = 2$  which would be needed for a local regularity result with  $R^{-3} \int_{Q(z_0, R)} |v|^2 dz$ . See for example [25].

<sup>3</sup>The finiteness of (1.2) implies  $\lim_{R \rightarrow 0} \int_{Q(z_0, R)} |v|^5 dz = 0$ , which gives us a small quantity. The finiteness of (1.3) implies (for the solutions of the equation)  $\lim_{R \rightarrow 0} \int_{B(x_0, R)} |v(x, t_0)|^3 dx = 0$ . This again gives a small quantity, but in this case it is not easy to exploit it, since we essentially have to show that some regularity propagates

In this paper we study local regularity results for axi-symmetric solutions of the 3D Navier-Stokes under an assumption that a weakened version of quantity (1.7) or, respectively, (1.8) is finite (but not necessarily small). These studies can be thought of as a continuation of the work started in [3], [10], and [4]. The exact assumption which we will use to replace (1.8), in the axi-symmetric situation, with the  $x_3$ -axis as the axis of symmetry, is

$$\operatorname{ess\,sup}_{(x,t) \in Q(z_0, R)} \sqrt{x_1^2 + x_2^2} |\bar{v}(x, t)| < +\infty \quad (1.9)$$

for some  $R > 0$ , where  $z_0$  lies on the  $x_3$ -axis and we denote by  $\bar{v}(x, t)$  the projection of the velocity vector  $v(x, t)$  into the plane passing through  $x$  and the axis of symmetry  $x_3$ . Similarly, the exact assumption which will replace (1.7) in the axi-symmetric situation, with the  $x_3$ -axis as the axis of symmetry, is

$$\operatorname{ess\,sup}_{(x,t) \in Q(z_0, R)} \sqrt{t_0 - t} |\bar{v}(x, t)| < +\infty \quad (1.10)$$

for some  $R > 0$ , where  $z_0$  and  $\bar{v}$  are as above. Our main results are as follows.

**Theorem 1.1.** *Assume that  $v \in L_3(Q(z_0, R))$  is an axially symmetric weak solution to the Navier-Stokes equations in  $Q(z_0, R)$  such that there exists an associated pressure field  $q \in L_{\frac{3}{2}}(Q(z_0, R))$ . If, in addition,  $v$  satisfies (1.10), then  $z_0$  is a regular point of  $v$ .*

**Theorem 1.2.** *Assume that  $v \in L_3(Q(z_0, R))$  is an axially symmetric weak solution to the Navier-Stokes equations in  $Q(z_0, R)$  such that there exists an associated pressure field  $q \in L_{\frac{3}{2}}(Q(z_0, R))$ . Suppose that  $v$  is essentially bounded in the space-time cylinders of the form  $B(x_0, R) \times ]t_0 - R^2, t'[$  for each  $t' < t_0$ , where the bound may depend on  $t'$ . If, in addition,  $v$  satisfies (1.9), then  $z_0$  is a regular point of  $v$ .*

These are local versions of the main results in the paper [10]. Similar (but not identical) results also appeared in [3] and [4].

For completeness, we formulate another theorem, which is a local version of the corresponding global regularity result in [11] and [30].

**Theorem 1.3.** *Assume that  $v \in L_3(Q(z_0, R))$  is an axially symmetric weak solution to the Navier-Stokes equations in  $Q(z_0, R)$  such that there exists*

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backwards in time.

an associated pressure field  $q \in L_{\frac{3}{2}}(Q(z_0, R))$ . Suppose that  $v$  is essentially bounded in the space-time cylinders of the form  $B(x_0, R) \times ]t_0 - R^2, t'[$  for each  $t' < t_0$ , where the bound may depend on  $t'$ . If, in addition, the field  $v$  has no swirl, i. e.  $v = \bar{v}$ , then  $z_0$  is a regular point of  $v$ .

On a conceptual level our method will be close to the one used in [10], and will rely on the Liouville-type theorems established in that paper. However, certain important technical parts will be treated in a different way.

We first recall some terminology related to the Liouville-type results for the Navier-Stokes proved in [10]. An *ancient solution* of the Navier-Stokes equation is a solution defined in  $\mathbb{R}^n \times ]-\infty, 0[$ . We are interested in ancient solutions with bounded velocity, see Definition 2.3. Non-zero solutions of this form can be generated by a natural re-scaling and limiting procedures at a potential singularity, see Section 2. The definition of the ancient solutions still allows for the “parasitic solutions” of the form  $u(x, t) = b(t)$  (for any bounded  $b: ]-\infty, 0[ \rightarrow \mathbb{R}^n$ ), with the corresponding pressure  $p$  given by  $p(x, t) = -b'(t) \cdot x$ , see Remark 2.6. To exclude these solutions (which - under some natural assumptions - cannot arise from the re-scaling procedures, see Theorem 2.8) we introduce the notion of the ancient mild solutions. These are the ancient solutions which satisfy the natural representation formula

$$u(t) = S(t - t_0)u(t_0) + \int_{t_0}^t (S(t - s) P \operatorname{div} (u(s) \otimes u(s))) ds \quad (1.11)$$

for some sequence of times  $t_0 \rightarrow -\infty$ , where  $S$  is the solution operator for the heat equation and  $P$  is the Helmholtz projection onto the div-free fields. (We remark that the usual integration by parts shows that the integral on the left-hand side is well defined for  $u \in L_\infty$ .) See [10] for details. The strongest conjecture regarding the Liouville-type results one can make about the Navier-Stokes equations is the following:

*Conjecture (L): The velocity field of any bounded mild ancient solution of the Navier-Stokes equations is constant.*

The conjecture was proved for  $n = 2$  and also for axi-symmetric solutions in 3D, provided the additional decay condition

$$\sqrt{x_1^2 + x_2^2} |v(x, t)| \leq C \quad \text{in } \mathbb{R}^3 \times ]-\infty, 0[ \quad (1.12)$$

is satisfied, see [10].

What would be the implications of the validity of Conjecture (L) for the regularity theory? Roughly speaking, if Conjecture (L) is valid, then all the problems discussed above concerning regularity in the presence of a scale-invariant estimates are solved. Indeed, the re-scaling procedure preserves any scale-invariant estimate, and typically the estimate will also be preserved in the limiting process. Therefore as a result of the re-scaling we get, in the limit, a non-zero bounded mild ancient solution for which a scale invariant quantity is finite. Conjecture (L) would leave only one candidate for the mild ancient solution - namely a non-zero constant velocity field. However, this possibility is typically not compatible with a finite scale-invariant bound.

We can summarize the above as follows:

$$\begin{array}{c} \text{scale invariant} \\ \text{estimate} \end{array} + \text{Conjecture (L)} \Rightarrow \text{regularity} .$$

Singularities for which some scale-invariant quantity is bounded are often called Type I singularities. (The most common definition of Type I singularities uses quantity (1.7).)

While we do not really know what the likelihood of Conjecture (L) being true is for the general 3D solutions, we are quite confident that the conjecture is indeed true for the axi-symmetric solutions. The axi-symmetric case of Conjecture (L) would imply much stronger results than Theorems 1.1 and 1.2 above. However, we have not been able to fully prove Conjecture (L) in the axi-symmetric case so far.

Our method of proof of Theorems 1.1 and 1.2 is can be described as follows. Roughly speaking, we will show that, on the solutions of the equations, the assumed scale invariant bounds imply that all the other important scale-invariant quantities are bounded, and these bounds, together with the known partial regularity theory ([1, 12, 17]), lead relatively easily to the bounds required by the Liouville theorems in [10].

The idea that a bound of one scale-invariant quantity should lead (for the solutions of the equations) to bounds on other scale invariant quantities is of course not new. However, examples from some other elliptic/parabolic PDEs show that these issues can be subtle. For example, in the theory of harmonic mappings or the harmonic map heat flow we do have a scale-invariant a-priori bound, which corresponds to a bound of quantity (1.5). However, it is known that singularities can still arise, and therefore the bound corresponding to (1.2) (which is known to imply regularity in that situation) cannot be derived from (the analogue of) (1.5).

We now informally explain the main steps of the proof. One is that the swirl component of the velocity field,  $v_\varphi = v \cdot e_\varphi$  satisfies a scalar parabolic equation which enables one to gain some regularity. To explain this, we need to introduce the following simple notation. Let  $e_1, e_2, e_3$  be an orthogonal basis of the Cartesian coordinates  $x_1, x_2, x_3$  and  $e_\varrho, e_\varphi, e_3$  be an orthogonal basis of the cylindrical coordinates  $\varrho, \varphi, x_3$  chosen so that

$$e_\varrho = \cos \varphi e_1 + \sin \varphi e_2, \quad e_\varphi = -\sin \varphi e_1 + \cos \varphi e_2, \quad e_3 = e_3.$$

Then, for any vector-valued field  $v$ , we have representations

$$v = v_i e_i = v_1 e_1 + v_2 e_2 + v_3 e_3 = v_\varrho e_\varrho + v_\varphi e_\varphi + v_3 e_3.$$

Next, letting  $f = \varrho v_\varphi$ , we have

$$\partial_t f + \bar{v} \cdot \nabla f = \Delta f - \frac{2}{\varrho} \frac{\partial f}{\partial \varrho}. \quad (1.13)$$

We would like to prove a  $L_\infty$ -bound on  $f$ . Such a bound will give us enough information about  $v - \bar{v}$  so that, oversimplifying slightly, we can replace  $\bar{v}$  by  $v$  in our assumptions. The  $L_\infty$  bound for (1.13) does not follow from general parabolic theory, since the general theory requires more regularity than we have. However, it is known that if the drift term in equations such as (1.13) is div-free, one can prove the  $L_\infty$  estimate for  $f$  with weaker assumptions on the coefficients. See for example [6] for the elliptic case and [32] for the parabolic case.

Another important step in the proof is conceptually the same as deriving an estimate for the quantities (1.4) and (1.6) from the boundedness of (1.7). This can be done by bootstrapping the energy inequality. This idea was used for example in [24]. The technical details are somewhat complicated, but the main idea can be explain at a heuristic level as follows. To simplify notation, we will use  $Q(R)$  for  $Q(0, R) = Q((0, 0), R)$ ,  $Q$  for  $Q(1)$ ,  $B(R)$  for  $B(0, R)$ , and  $B$  for  $B(1)$ .

We first note that (1.7) implies a bound on the (scale-invariant) quantity  $R^{1-\frac{2}{l}-\frac{3}{s}} \|v\|_{s,l,Q(R)}$  for  $l < 2$  and  $s \geq 1$ . Here,  $\|\cdot\|_{s,l,Q(R)}$  is the norm of the mixed Lebesgue space  $L_{s,l}(Q(R)) = L_l(-R^2, 0; L_s(B(R)))$ .

Let  $[u]_{Q(R)}$  denote the parabolic energy norm in  $Q(R)$ , i. e.

$$[u]_{Q(R)}^2 = \text{ess sup}_{t \in [-R^2, 0]} \|u(\cdot, t)\|_{L_2(B(R))}^2 + \|\nabla u\|_{L_2(Q(R))}^2.$$

To avoid technicalities, let us pretend that the pressure satisfies  $|q| \sim |v|^2$ . In reality it is not quite true and in the rigorous proof one has to deal with this, but the procedure is well understood. Therefore our simplifying assumption  $|q| \sim |v|^2$  is reasonable for the heuristics. We will now work with  $R = 1$ , but we can scale the calculations to any  $R > 0$ , if we divide all the involved quantities by the powers of  $R$  which make them scale-invariant.

The local energy inequality implies

$$[v]_Q^2 \lesssim \|v\|_{L_3(Q(2))}^3 + \|v\|_{L_3(Q(2))}^2. \quad (1.14)$$

We can now “bootstrap” this inequality. There are some technical complications coming from the fact that we have  $Q(2)$  on the right-hand side of (1.14) but only  $Q$  on the left-hand side. Such problems come up often in local regularity theory of elliptic and parabolic equations, and it is quite well-understood how to deal with them by suitable iteration procedures. Therefore, cheating slightly, we can pretend that we actually have  $Q$  on the right-hand side of inequality (1.14):

$$[u]_Q^2 \lesssim \|v\|_{L_3(Q)}^3 + \|v\|_{L_3(Q)}^2. \quad (1.15)$$

To bootstrap, we estimate

$$\|v\|_{L_3(Q)} \lesssim [v]_Q^\alpha \|v\|_{s,l,Q}^\beta \quad (1.16)$$

for suitable  $\alpha, \beta > 0$ ,  $\alpha + \beta = 1$ , and use this in inequality (1.15). We see that when  $\alpha < 2/3$ , we can estimate  $[v]_Q$  in terms of the norm  $\|v\|_{s,l,Q}$ . (The process also works for  $\alpha = 2/3$  provided  $\|v\|_{s,l}$  is sufficiently small.)

It remains to determine the correct exponents  $\alpha, \beta$  in (1.16). Denoting by  $2^*$  the Sobolev exponent of the space  $W^{1,2}$  (i. e.  $1/2^* = 1/2 - 1/n$ ), we have by the Hölder inequality

$$\|v\|_{3,3} \leq \|v\|_{2,\infty}^{\alpha_1} \|v\|_{2^*,2}^{\alpha_2} \|v\|_{s,l}^{\alpha_3}, \quad (1.17)$$

where  $\alpha_1, \alpha_2, \alpha_3$  are non-negative numbers satisfying

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 1 \\ \frac{\alpha_1}{2} + \frac{\alpha_2}{2^*} + \frac{\alpha_3}{s} &= \frac{1}{3} \\ \frac{\alpha_2}{2} + \frac{\alpha_3}{l} &= \frac{1}{3} \end{aligned} \quad (1.18)$$

By Sobolev imbedding, we have from (1.17)

$$\|v\|_{L_3(Q)} \lesssim [v]_Q^{\alpha_1 + \alpha_2} \|v\|_{s,l,Q}^{\alpha_3}, \quad (1.19)$$

and we see that (1.16) holds true with  $\alpha = \alpha_1 + \alpha_2$ ,  $\beta = \alpha_3$ . Therefore the set of the parameters  $s, l$  for which the iteration procedure works is given by the condition that equations (1.18) for  $\alpha_1, \alpha_2, \alpha_3$  have a non-negative solution with  $\alpha_3 > 1/3$ . Solving (1.18) for  $n = 3$ , we obtain

$$\begin{aligned}\alpha_1 &= \frac{\frac{1}{s} + \frac{1}{l} - \frac{2}{3}}{\frac{3}{s} + \frac{2}{l} - \frac{3}{2}}, \\ \alpha_2 &= \frac{\frac{2}{s} + \frac{1}{l} - 1}{\frac{3}{s} + \frac{2}{l} - \frac{3}{2}}, \\ \alpha_3 &= \frac{1}{6\left(\frac{3}{s} + \frac{2}{l} - \frac{3}{2}\right)}.\end{aligned}$$

One can check easily that the conditions  $\alpha_1 \geq 0$ ,  $\alpha_2 \geq 0$ ,  $\alpha_3 > 1/3$  are equivalent to

$$\begin{aligned}\frac{1}{s} + \frac{1}{l} &\geq \frac{2}{3}, \\ \frac{2}{s} + \frac{1}{l} &\geq 1, \\ \frac{3}{s} + \frac{2}{l} &< 2.\end{aligned}\tag{1.20}$$

In the plane with coordinates  $x = \frac{1}{s}$  and  $y = \frac{1}{l}$ , the last set of equations describes a thin triangle contained in the first quadrant. It is easy to see that one can choose a suitable  $l < 2$  and  $s > 1$  for which these conditions are satisfied.

Above we worked with parabolic balls of radius of order 1. It is clear that the calculations can be “scaled” to  $Q(R)$  and  $Q(2R)$  if we divide the  $L_{s,l}$ -norms by suitable powers of  $R$  to obtain scale invariant quantities. It is also clear that of the restrictions in (1.20), only the last one is crucial, since by using Hölder inequality one can always move to higher exponents, as long as the power in the correct scaling factor remains positive.

This finishes our explanation of the heuristics behind the second step of the proof. We did not mention one more complication. Since our main assumption involves only  $\bar{v}$  and the information about  $v - \bar{v}$  is obtained from equation (1.13), we have to use different one set of parameters  $l, s$  for the  $\bar{v}$  component of  $v$  and a different set for  $v - \bar{v}$ . However, this is a technical issue which does not change the heuristics.

Once we know that the scaled energy-type quantities are bounded, it is not difficult to derive the bounds which we need in the version of the Liouville conjecture for axi-symmetric solutions which was proved in [10].

Another aim of the paper is to give an alternative approach to certain technical issues arising in the study of bounded ancient solutions. In the approach here, we do not use the exact representation formulae which were used in [10]. It turned out to be quite convenient to describe differentiability properties of bounded ancient solutions in terms of certain “uniform” Lebesgue and Sobolev spaces, compare with [15]. We hope that both approaches are of interest, and complement each other.

## 2 Preliminaries

In this Section, we recall known definitions of (weak) solutions to the Navier-Stokes.

**Definition 2.1.** *A weak solution to the Navier-Stokes equations in a domain  $\mathcal{O} \subset \mathbb{R}^n \times ]t_1, t_2[$  is a divergence free vector-valued field  $v \in L_{2,loc}(\mathcal{O})$  satisfying*

$$\int_{\mathcal{O}} (v \cdot \partial_t w + v \otimes v : \nabla w + v \cdot \Delta w) dx dt = 0$$

for any solenoidal vector-valued field  $w \in C_0^\infty(\mathcal{O})$ .

An important family of weak solutions is given by  $v(x, t) = \nabla h(x, t)$  where  $h : \mathcal{O} \rightarrow \mathbb{R}$  satisfies  $\Delta h = 0$ . (Dependence on  $t$  can be arbitrary). This example shows that further assumptions are needed to obtain some regularity of solutions in the time direction.

Very often, we shall study local regularity of solutions to the Navier-Stokes equations in the unit parabolic ball  $Q = B \times ]-1, 0[$ , where  $B = B(1) = B(0, 1)$ . It is not a loss of generality because of the Navier-Stokes scaling.

In local analysis, the most reasonable object to study is so-called suitable weak solutions, introduced by Caffarelli-Kohn-Nirenberg in [1]. We are going to use a slightly simpler definition of F.-H. Lin in [17]

**Definition 2.2.** *The pair  $v$  and  $q$  is called a suitable weak solutions to the Navier-Stokes equations in  $Q$  if the following conditions are satisfied:*

$$v \in L_{2,\infty}(Q) \cap W_2^{1,0}(Q), \quad q \in L_{\frac{3}{2}}(Q);$$

*$v$  and  $q$  satisfy the Navier-Stokes equations in the sense of distributions;*

for a.a.  $t \in ]-1, 0[$ , the local energy inequality

$$\int_{\mathcal{C}} \varphi(x, t) |v(x, t)|^2 dx + 2 \int_{-1}^t \int_{\mathcal{C}} \varphi |\nabla v|^2 dx dt' \leq \int_{-1}^t \int_{\mathcal{C}} \left\{ |v|^2 (\Delta \varphi + \partial_t \varphi) + v \cdot \nabla \varphi (|v|^2 + 2q) \right\} dx dt'$$

holds for all non-negative cut-off functions  $\varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$  vanishing in a neighborhood of the parabolic boundary of  $Q$ .

Here, the following functional spaces have been used:

$$\begin{aligned} L_{s,l}(Q) &= L_l(-1, 0; L_s(B)), & L_s &= L_{s,s}, \\ W_{s,l}^{1,0}(Q) &= \{v, \nabla v \in L_{s,l}(Q)\}, & W_s^{1,0} &= W_{s,s}^{1,0}, \\ W_{s,l}^{2,1}(Q) &= \{v, \nabla v, \partial_t v, \nabla^2 v \in L_{s,l}(Q)\}, & W_s^{2,1} &= W_{s,s}^{2,1}. \end{aligned}$$

The norm of the space  $L_{s,l}(Q)$  is denoted by  $\|\cdot\|_{s,l,Q}$ .

For further discussions of Definition 2.2, we refer the reader to papers [12] and [23].

In what follows, we shall assume  $v$  and  $q$  satisfy the following standing conditions:

pair  $v \in L_3(Q)$  and  $q \in L_{\frac{3}{2}}(Q)$  satisfies the Navier-Stokes equations

$$\text{in the sense of distributions;} \tag{2.1}$$

$$v \in L_\infty(B \times ]-1, -a^2[) \quad \forall a \in ]0, 1[; \tag{2.2}$$

$$\text{there is a number } 0 < r_1 < 1 \text{ such that } v \in L_\infty(Q_1), \tag{2.3}$$

where  $Q_1 = B_1 \times ]-1, 0[$ ,  $B_1 = \{r_1 < |x| < 1\}$ . To explain why there is no loss of generality, we first notice that the pair  $v$  and  $q$ , satisfying conditions (2.1)–(2.3), is in fact a suitable weak solution to the Navier-Stokes equations in  $Q$ . It is certainly true in  $B \times ]-1, -a^2[$  but condition (2.1) allows us to extend this property to the whole cylinder  $Q$ .

Consider now an arbitrary suitable weak solution  $v$  and  $q$  in  $Q$ . Let  $S \subset B \times ]-1, 0[$  be a set of singular points of  $v$ . It is closed in  $Q$ . As it was shown in [1],  $\mathcal{P}^1(S) = 0$ , where  $\mathcal{P}^1$  is the one-dimensional parabolic Hausdorff measure. Let us assume that  $S \neq \emptyset$ . We can choose number  $R_1$

and  $R_2$  satisfying  $0 < R_1 < R_2 < 1$  such that  $S \cap \overline{Q(R_1) \setminus Q(R_2)} = \emptyset$  and  $S \cap B(R_2) \times ] - R_2^2, 0] \neq \emptyset$ . We put

$$t_0 = \inf\{t : (x, t) \in S \cap B(R_2) \times ] - R_2^2, 0]\}.$$

Clearly,  $(x_0, t_0) \in S$  for some  $x_0 \in B(R_2)$ . In a sense,  $t_0$  is the first singular time of our suitable weak solution  $v$  and  $q$  in  $Q(R_1)$ . Next, the one-dimensional Hausdorff measure of the set

$$S_{t_0} = \{x_* \in B(R_2) : (x_*, t_0) \text{ is a singular point}\}$$

is zero as well. Therefore, given  $x_0 \in S_{t_0}$ , we can find sufficiently small  $0 < r < \sqrt{R_2^2 + t_0}$  such that  $B(x_0, r) \Subset B(R_2)$  and  $\partial B(x_0, r) \cap S_{t_0} = \emptyset$ . Since the velocity field  $v$  is Hölder continuous at regular points, we can ensure that all conditions of type (2.1)–(2.3) hold in the parabolic ball  $Q(z_0, r)$  with  $z_0 = (x_0, t_0)$ . We may shift and re-scale our solution if  $x_0 = 0$  and  $r \neq 1$ .

In our investigations of regularity of suitable weak solutions, the particular case of weak solutions, see Definition 2.1, plays a crucial role. Here is the corresponding definition.

**Definition 2.3.** ([10]) *A bounded divergence free field  $u \in L_\infty(Q_-; \mathbb{R}^n)$  is called a weak bounded ancient solution (or simply bounded ancient solution) to the Navier-Stokes equations if*

$$\int_{Q_-} (u \cdot \partial_t w + u \otimes u : \nabla w + u \cdot \Delta w) dz = 0$$

for any  $w \in \mathring{C}_0^\infty(Q_-)$ .

Here, we have used the following notation:

$$Q_- \equiv \mathbb{R}^n \times ] - \infty, 0[, \quad \mathring{C}_0^\infty(Q_-) = \{v \in C_0^\infty(Q_-), \quad \operatorname{div} v = 0\}.$$

The notion of bounded ancient solutions is not quite satisfactory for our purposes since it allows “parasitic solutions”  $u(x, t) = b(t)$ , where  $b$  is an  $L_\infty$ -function. (These correspond to harmonic function  $h(x, t) = b(t) \cdot x$ .)

The important subclass of bounded ancient solution was introduced in [10]. It consists of the so-called mild bounded ancient solutions, i.e., bounded ancient solution satisfying the representation formula (1.11). In [10], we also

showed that one has a natural decomposition  $u(x, t) = w(x, t) + b(t)$ , where  $w$  is given by the right hand side of (1.11) on a suitable interval  $]t_1, t_*[$ . (In particular,  $w$  is Hölder continuous.)

In this paper, we give another proof of the decomposition of arbitrary bounded ancient solutions into regular and singular parts, see Section 5. It is based on recovery of a pressure field associated with a given bounded ancient solution. To formulate our theorem about the pressure, we need to introduce certain functional spaces

By  $L_m(\Omega)$  and  $W_m^1(\Omega)$ , we denote the usual Lebesgue and Sobolev spaces of functions defined on  $\Omega \in \mathbb{R}^n$ . We also need parabolic versions of Sobolev's spaces:

$$W_m^{1,0}(Q_T) = \{|v| + |\nabla v| \in L_m(Q_T)\},$$

$$W_m^{2,1}(Q_T) = \{|v| + |\nabla v| + |\partial_t v| + |\nabla^2 v| \in L_m(Q_T)\}$$

where  $Q_T = \Omega \times ]0, T[$ . The norm of the space  $L_m(\Omega)$  is denoted by  $\|\cdot\|_{m,\Omega}$ .

We also going to use the following "uniform" spaces (compare with [15]):

$$\mathcal{L}_m(Q_-) = \{\|f\|_{\mathcal{L}_m(Q_-)} = \sup_{z_0 \in Q_-} \|f\|_{m,Q(z_0,1)} < +\infty\},$$

$$\mathcal{W}_m^{1,0}(Q_-) = \{\|f\|_{\mathcal{W}_m^{1,0}(Q_-)} = \sup_{z_0 \in Q_-} \|f\|_{W_m^{1,0}(Q(z_0,1))} < +\infty\},$$

$$\mathcal{W}_m^{2,1}(Q_-) = \{\|f\|_{\mathcal{W}_m^{2,1}(Q_-)} = \sup_{z_0 \in Q_-} \|f\|_{W_m^{2,1}(Q(z_0,1))} < +\infty\}.$$

To define the regular part of the pressure, we recall the known fact, see e.g. [28]. Given  $F = L_\infty(\mathbb{R}^n; \mathbb{M}^{n \times n})$ , there exists a unique function  $p_F \in BMO(\mathbb{R}^n)$  such that  $[p_F]_{B(1)} = 0$  ( $[f]_\Omega$  is the mean value of a function  $f$  over a spatial domain  $\Omega \in \mathbb{R}^n$ ) and

$$\Delta p_F = -\operatorname{div} \operatorname{div} F = F_{ij,ij} \quad \text{in } \mathbb{R}^3$$

in the sense of distributions. Moreover, function  $p_F$  meets the estimate

$$\|p_F\|_{BMO(\mathbb{R}^n)} \leq c(n) \|F\|_{\infty, \mathbb{R}^n}.$$

So, given a bounded ancient solution  $u$ , we define a regular part of the pressure  $p_{u \otimes u}$ .

**Theorem 2.4.** *Let  $u$  be an arbitrary bounded ancient solution. For any number  $m > 1$ ,*

$$|\nabla u| + |\nabla^2 u| + |\nabla p_{u \otimes u}| \in \mathcal{L}_m(Q_-).$$

*In addition, for each  $t_0 \leq 0$ , there exists a function  $b_{t_0} \in L_\infty(t_0 - 1, t_0)$  with the following property*

$$\sup_{t_0 \leq 0} \|b_{t_0}\|_{L_\infty(t_0-1, t_0)} \leq c(n) < +\infty.$$

*If we let  $u^{t_0}(x, t) = u(x, t) + b_{t_0}(t)$ ,  $(x, t) \in Q^{t_0} = \mathbb{R}^n \times ]t_0 - 1, t_0[$ , then, for any number  $m > 1$ ,*

$$\sup_{z_0=(x_0, t_0), x_0 \in \mathbb{R}^n, t_0 \leq 0} \|u^{t_0}\|_{W_m^{2,1}(Q(z_0, 1))} \leq c(m, n) < +\infty.$$

*Moreover, for each  $t_0 \leq 0$ , functions  $u$  and  $u^{t_0}$  obey the system of equations*

$$\partial_t u^{t_0} + \operatorname{div} u \otimes u - \Delta u = -\nabla p_{u \otimes u}, \quad \operatorname{div} u = 0$$

*a.e in  $Q^{t_0}$ .*

**Remark 2.5.** *The first equation of the latter system can be reduced to the form*

$$\partial_t u + \operatorname{div} u \otimes u - \Delta u = -\nabla p_{u \otimes u} - b'_{t_0} \quad \text{in } Q^{t_0},$$

*which is understood in the sense of distributions,  $b'_{t_0}(t) = db_{t_0}(t)/dt$ . So, the real pressure field in  $Q^{t_0}$  is the distribution  $p_{u \otimes u} + b'_{t_0} \cdot x$ .*

**Remark 2.6.** *We can find a measurable vector-valued function  $b$  defined on  $] - \infty, 0[$  and having the following property. For any  $t_0 \leq 0$ , there exists a constant vector  $c_{t_0}$  such that*

$$\sup_{t_0 \leq 0} \|b - c_{t_0}\|_{L_\infty(t_0-1, t_0)} < +\infty.$$

*Moreover, the Navier-Stokes system takes the form*

$$\partial_t u + \operatorname{div} u \otimes u - \Delta u = -\nabla(p_{u \otimes u} + b' \cdot x), \quad \operatorname{div} u = 0$$

*in  $Q_-$  in the sense of distributions.*

**Remark 2.7.** *Bounded ancient solutions with  $b' = 0$  were introduced in [10] and called mild ancient solutions. They were systematically studied in [10] and in particular it was shown there that mild ancient solutions are infinitely smooth.*

Our interest in bounded ancient solutions comes from their appearance as natural limits of suitable re-scaling procedures at potential singularities. In the context of solutions to the Navier-Stokes equations in  $\mathbb{R}^n \times ]t_1, t_2[$ , this was studied in [10].

We shall now show that re-scaling procedure also works for potential singularities of local suitable weak solutions. The important point will be that, even in the local (in space) situation, the solutions arising from the re-scaling procedure at a potential singularity are still mild bounded ancient solutions. To be more precise, let us consider local solutions of the Navier-Stokes equations satisfying assumptions (2.1)–(2.3) and introduce functions

$$G(t) = \max_{x \in \overline{B}(r_1)} |v(x, t)|, \quad M(t) = \sup_{-1 \leq \tau \leq t} G(\tau).$$

Assume that there are singular points of  $v$  which are located somewhere on the set  $\{(x, 0) : |x| \leq r_1\}$ . By known regularity criteria (see e.g. [25]), we have

$$M(t) > \frac{\varepsilon}{\sqrt{-t}}$$

for some  $\varepsilon > 0$  and thus

$$M(t) \rightarrow +\infty$$

if  $t \rightarrow 0-$ . We can construct a sequence  $t_k$  such that  $t_k \in ]-1, 0[$ ,  $t_k < t_{k+1}$ ,  $t_k \rightarrow 0$ , and

$$M(t_k) = G(t_k) = |v(x_k, t_k)| \rightarrow +\infty$$

for some  $x_k \in \overline{B}(r_1)$ .

Next, we scale  $v$  and  $q$  the following way

$$u^k(y, s) = \lambda_k v(x, t), \quad p^k(y, s) = \lambda_k^2 q(x, t),$$

where  $x = x^k + \lambda_k y$ ,  $t = t_k + \lambda_k^2 s$ , and  $\lambda_k = 1/M_k$ . The ball  $B(r)$  is mapped by the change of variables onto  $B(-x^k/\lambda_k, r/\lambda_k)$  and if  $r \in ]r_2, 1]$  then, given any  $R > 0$ ,

$$B(R) \subset B\left(-\frac{x^k}{\lambda_k}, \frac{r}{\lambda_k}\right)$$

for sufficiently large  $k$ . For scaled functions  $u^k$  and  $p^k$ , we know

$u^k$  and  $p^k$  satisfy the Navier-Stokes equations and

$$|u^k| \leq 1 \quad \text{in } B\left(-\frac{x^k}{\lambda_k}, \frac{1}{\lambda_k}\right) \times \left]-\frac{1-t_k}{\lambda_k^2}, 0\right[; \quad (2.4)$$

$$|u^k(0,0)| = 1. \quad (2.5)$$

**Theorem 2.8.** *For each  $a > 0$ , the sequence  $\{u^k\}$  is uniformly Hölder continuous on the closure of  $Q(a)$  for sufficiently large  $k$  and a subsequence  $\{u^{k_j}\}$  of  $\{u^k\}$  converges uniformly on compact subsets of  $\mathbb{R}^n \times ]-\infty, 0]$  to a mild bounded ancient solution  $u$  with  $|u(0,0)| = 1$ .*

**Remark 2.9.** *The main point of the theorem is that the limit  $u$  is a mild solution, i.e., the parasitic solutions cannot appear in this re-scaling procedure.*

**PROOF OF THEOREM 2.8** Our solution  $v$  and  $q$  has good properties inside  $Q_1$ . Let us enumerate them. Let  $Q_2 = B_2 \times ]-\tau_2^2, 0[$ , where  $0 < \tau_2 < 1$ ,  $B_2 = \{r_1 < r_2 < |x| < a_2 < 1\}$ . Then, for any natural  $k$ ,

$$z = (x, t) \mapsto \nabla^k v(z) \text{ is Hölder continuous in } \overline{Q_2};$$

$$q \in L_{\frac{3}{2}}(-\tau_2^2, 0; C^k(\overline{Q_2})).$$

The corresponding norms are estimated by constants depending on  $\|v\|_{3,Q}$ ,  $\|q\|_{\frac{3}{2},Q}$ ,  $\|v\|_{\infty,Q_1}$ , and numbers  $k$ ,  $r_1$ ,  $r_2$ ,  $a_2$ ,  $\tau_2$ . In particular, we have

$$\max_{x \in \overline{B_2}} \int_{-\tau_2^2}^0 |\nabla q(x, t)|^{\frac{3}{2}} dt \leq c_1 < \infty. \quad (2.6)$$

Proof of this statements can be done by induction and founded in [5], [12], and [18].

Now, let us decompose the pressure  $q = q_1 + q_2$ . For  $q_1$ , we have

$$\Delta q_1(x, t) = -\operatorname{div} \operatorname{div} \left[ \chi_B(x) v(x, t) \otimes v(x, t) \right], \quad x \in \mathbb{R}^3, \quad -1 < \tau < 0,$$

where  $\chi_B(x) = 1$  if  $x \in B$  and  $\chi_B(x) = 0$  if  $x \notin B$ . Obviously, the estimate

$$\int_{-1}^0 \int_{\mathbb{R}^3} |q_1(x, t)|^{\frac{3}{2}} dx dt \leq c \int_Q |v|^3 dz$$

holds which is a starting point for local regularity of  $q_1$ . Using essentially the same bootstrap arguments, we can show

$$\max_{x \in \overline{B_3}} \int_{-\tau_2^2}^0 |\nabla q_1(x, t)|^{\frac{3}{2}} dt \leq c_2 < \infty, \quad (2.7)$$

where  $B = \{r_2 < r_3 < |x| < a_3 < a_2\}$ . From (2.6) and (2.7), it follows that

$$\max_{x \in \overline{B_3}} \int_{-\tau_2^2}^0 |\nabla q_2(x, t)|^{\frac{3}{2}} dt \leq c_3 < \infty. \quad (2.8)$$

But clearly  $q_2$  is a harmonic function in  $B$ , thus, by the maximum principle, we have

$$\max_{x \in \overline{B}(r_4)} \int_{-\tau_2^2}^0 |\nabla q_2(x, t)|^{\frac{3}{2}} dt \leq c_3 < \infty, \quad (2.9)$$

where  $r_4 = (r_3 + a_3)/2$ .

Let us re-scale each part of the pressure separately, i.e.,

$$p_i^k(y, s) = \lambda_k^2 q_i(x, t), \quad i = 1, 2,$$

so that  $p^k = p_1^k + p_2^k$ . As it follows from (2.9), for  $p_2^k$ , we have

$$\sup_{y \in B(-x^k/\lambda_k, r_4/\lambda_k)} \int_{-(\tau_2^2 - t_k)/\lambda_k^2}^0 |\nabla_y p_2^k(y, s)|^{\frac{3}{2}} ds \leq c_3 \lambda_k^{\frac{5}{2}}. \quad (2.10)$$

The first component of the pressure satisfies the equation

$$\Delta_y p_1^k(y, s) = -\operatorname{div}_y \operatorname{div}_y (\chi_{B(-x^k/\lambda_k, 1/\lambda_k)}(y) u^k(y, s) \otimes u^k(y, s)), \quad y \in \mathbb{R}^3,$$

for all possible values of  $s$ . For such a function, we have the standard estimate

$$\|p_1^k(\cdot, s)\|_{BMO(\mathbb{R}^3)} \leq c \quad (2.11)$$

for all  $s \in ] - (1 - t_k)/\lambda_k^2, 0[$ .

We slightly change  $p_1^k$  and  $p_2^k$  setting

$$\overline{p}_1^k(y, s) = p_1^k(y, s) - [p_1^k]_{B(1)}(s) \quad \overline{p}_2^k(y, s) = p_2^k(y, s) - [p_2^k]_{B(1)}(s)$$

so that  $[\bar{p}_1^k]_{B(1)}(s) = 0$  and  $[\bar{p}_2^k]_{B(1)}(s) = 0$ .

Now, we pick up an arbitrary positive number  $a$  and fix it. Then from (2.10) and (2.11) it follows that for sufficiently large  $k$  we have

$$\int_{Q(a)} |\bar{p}_1^k|^{\frac{3}{2}} de + \int_{Q(a)} |\bar{p}_2^k|^{\frac{3}{2}} de \leq c_4(c_2, c_3, a).$$

Using the same bootstrap arguments, we can show that the following estimate is valid:

$$\|u^k\|_{C^\alpha(\bar{Q}(a/2))} \leq c_5(c_2, c_3, c_4, a)$$

for some positive number  $\alpha < 1/3$ . Indeed, the norm  $\|u^k\|_{C^\alpha(\bar{Q}(a/2))}$  is estimated with the help of norms  $\|u^k\|_{L^\infty(Q(a))}$  and  $\|\bar{p}^k\|_{L^{\frac{3}{2}}(Q(a))}$ , where  $\bar{p}^k = \bar{p}_1^k + \bar{p}_2^k$ . Hence, using the diagonal Cantor procedure, we can select subsequences such that for some positive  $\alpha$  and for any positive  $a$

$$\begin{aligned} u^k &\rightarrow u && \text{in } C^\alpha(Q(a)), \\ \bar{p}_1^k &\rightarrow \bar{p}_1, && \text{in } L^{\frac{3}{2}}(Q(a)), \quad [\bar{p}_1]_{B(1)}(s) = 0, \\ \bar{p}_2^k &\rightarrow \bar{p}_2 && \text{in } L^{\frac{3}{2}}(Q(a)), \quad [\bar{p}_2]_{B(1)}(s) = 0. \end{aligned}$$

Moreover,  $u$  is a bounded ancient solution with the total pressure  $\bar{p} = \bar{p}_1 + \bar{p}_2$ , where  $\bar{p}_1 = p_{u \otimes u}$ .

Next, for sufficiently large  $k$ , we get from (2.10) that

$$\sup_{y \in \bar{B}(a)} \int_{Q(a)} |\nabla p_2^k(y, s)|^{\frac{3}{2}} ds \leq c_3 \lambda_k^{\frac{5}{2}}.$$

Hence,  $\nabla p_2 = 0$  in  $Q(a)$  for any  $a > 0$ . So,  $p_2(y, s)$  is identically zero. This allows us to conclude that the pair  $u$  and  $p_{u \otimes u}$  is a solution to the Navier-Stokes equations in the sense of distributions and thus  $u$  is a nontrivial mild bounded ancient solution satisfying the condition  $|u(0, 0)| = 1$ . Theorem 2.8 is proved.

### 3 Axially Symmetric Suitable Weak Solutions

Without loss of generality, the problem of local regularity of weak solutions (not necessary being axially symmetric) to the Navier-Stokes equations can

be formulated as follows. Let us consider a pair of functions  $v \in L_3(Q)$  and  $q \in L_{\frac{3}{2}}(Q)$ , defined in the unit space-time cylinder  $Q = \mathcal{C} \times ]-1, 0[$ , where  $\mathcal{C} = \{x = (x', x_3), x' = (x_1, x_2), |x'| < 1, |x_3| < 1\}$  is the unit spatial cylinder of  $\mathbb{R}^3$ , which satisfies the Navier-Stokes system in  $Q$  in the sense of distributions. The question we are interested in is under what additional conditions on  $v$  and  $q$ , the space-time origin  $z = (x, t) = 0$  is a regular point of  $v$ . By the definition, the velocity  $v$  is regular at the point  $z = 0$  if there exists a positive number  $r$  such that  $v$  is essentially bounded in the space-time cylinder  $Q(r)$ . Here,  $Q(r) = \mathcal{C}(r) \times ]-r^2, 0[$  and  $\mathcal{C}(r) = \{|x'| < r, |x_3| < r\}$ . In contrast to traditional setting, we replace the usual balls with cylinders, which is quite convenient in the case of axial symmetry. As usual, we set

$$\bar{v} = v_\varrho e_\varrho + v_3 e_3 \quad \widehat{v} = v_\varphi e_\varphi$$

for  $v = v_\varrho e_\varrho + v_\varphi e_\varphi + v_3 e_3$ .

We reformulate our main results for these canonical domains. The general case is obtained by re-scaling.

**Theorem 3.1.** *Assume that functions  $v \in L_3(Q)$  and  $q \in L_{\frac{3}{2}}(Q)$  are an axially symmetric weak solution to the Navier-Stokes equations in  $Q$ . Let, in addition, for some positive constant  $C$ ,*

$$|\bar{v}(x, t)| \leq \frac{C}{\sqrt{-t}} \quad (3.1)$$

for almost all points  $z = (x, t) \in Q$ . Then  $z = 0$  is a regular point of  $v$ .

**Theorem 3.2.** *Assume that functions  $v \in L_3(Q)$  and  $q \in L_{\frac{3}{2}}(Q)$  are an axially symmetric weak solution to the Navier-Stokes equations in  $Q$ . Let, in addition,*

$$v \in L_\infty(\mathcal{C} \times ]-1, -a^2[) \quad (3.2)$$

for each  $0 < a < 1$  and

$$|\bar{v}(x, t)| \leq \frac{C}{|x'|} \quad (3.3)$$

for almost all points  $z = (x, t) \in Q$  with some positive constant  $C$ . Then  $z = 0$  is a regular point of  $v$ .

It is well-known due to Caffarelli-Kohn-Nirenberg that if  $z = (x, t)$  is singular (i.e., not regular) point of  $v$ , then there must be  $x' = 0$ . In other words, all singular points must belong to the axis of symmetry which is axis  $x_3$ .

**Lemma 3.3.** *Assume that functions  $v \in L_3(Q)$  and  $q \in L_{\frac{3}{2}}(Q)$  are an axially symmetric weak solution to the Navier-Stokes equations in  $Q$ . Let, in addition, condition (3.2) hold. Then following estimate is valid:*

$$\operatorname{ess\,sup}_{z \in Q(1/2)} |\varrho v_\varphi(z)| \leq C(M) \left( \int_{Q(3/4)} |\varrho v_\varphi|^{\frac{10}{3}} dz \right)^{\frac{3}{10}}, \quad (3.4)$$

where

$$M = \left( \int_{Q(3/4)} |\bar{v}|^{\frac{10}{3}} dz \right)^{\frac{3}{10}} + 1.$$

For the reader convenience, we put the proof of Lemma 3.3 in Appendix II, see also [2] and [3]. Here, we would like to notice the following.

**Remark 3.4.** *Under the assumptions of Lemma 3.3, the pair  $v$  and  $q$  forms a suitable weak solution to the Navier-Stokes equations in  $Q$ . Hence, the right hand side of (3.4) is bounded from above.*

We recall that the Navier-Stokes equations are invariant with respect to the following scaling:

$$u(x, t) = \lambda v(\lambda x, \lambda^2 t), \quad p(x, t) = \lambda^2 q(\lambda x, \lambda^2 t)$$

So, new functions  $u$  and  $p$  satisfy the Navier-Stokes equations in a suitable domain.

With some additional notation

$$\begin{aligned} \mathcal{C}(x_0, R) &= \{x \in \mathbb{R}^3 \mid x = (x', x_3), \ x' = (x_1, x_2), \\ &|x' - x'_0| < R, \ |x_3 - x_{03}| < R\}, \quad \mathcal{C}(R) = \mathcal{C}(0, R), \quad \mathcal{C} = \mathcal{C}(1); \\ z = (x, t), \quad z_0 = (x_0, t_0), \quad Q(z_0, R) &= \mathcal{C}(x_0, R) \times ]t_0 - R^2, t_0[, \\ Q(R) &= Q(0, R), \quad Q = Q(1), \end{aligned}$$

we introduce certain scale-invariant functionals:

$$\begin{aligned} A(z_0, r; v) &= \operatorname{ess\,sup}_{t_0 - r^2 < t < t_0} \frac{1}{r} \int_{\mathcal{C}(x_0, r)} |v(x, t)|^2 dx, \\ E(z_0, r; v) &= \frac{1}{r} \int_{Q(z_0, r)} |\nabla v|^2 dz, \quad D(z_0, r; q) = \frac{1}{r^2} \int_{Q(z_0, r)} |q|^{\frac{3}{2}} dz, \end{aligned}$$

$$C(z_0, r; v) = \frac{1}{r^2} \int_{Q(z_0, r)} |v|^3 dz, \quad H(z_0, r; v) = \frac{1}{r^3} \int_{Q(z_0, r)} |v|^2 dz,$$

$$M_{s,l}(z_0, r; v) = \frac{1}{r^\kappa} \int_{t_0-r^2}^{t_0} \left( \int_{C(x_0, r)} |v|^s dx \right)^{\frac{l}{s}} dt,$$

where  $\kappa = l(\frac{3}{s} + \frac{2}{l} - 1)$  and  $s \geq 1, l \geq 1$ . As it was shown in [25], the following inequality holds

$$C(z_0, r; f) \leq cA^\mu(z_0, r; f)(M_{s,l}(z_0, r; f))^{\frac{1}{m}}(E(z_0, r; f) + H(z_0, r; f))^{\frac{m-1}{m}}, \quad (3.5)$$

where

$$\mu = \frac{l}{m} \left( \frac{3}{s} + \frac{3}{l} - 2 \right), \quad m = 2l \left( \frac{3}{s} + \frac{2}{l} - \frac{3}{2} \right),$$

provided

$$\frac{3}{s} + \frac{2}{l} - \frac{3}{2} \geq \max \left\{ \frac{1}{2} - \frac{1}{s}, \frac{1}{s} - \frac{1}{6} \right\}. \quad (3.6)$$

Actually, inequality (3.5) is but the result of application of Hölder's inequality and special Galiardo-Nireberg's inequality.

The essential technical part of the proof of Theorem 3.1 is the following lemma.

**Lemma 3.5.** *Under assumptions of Theorem 3.1, we have the estimate*

$$A(z_b, r; v) + E(z_b, r; v) + C(z_b, r; v) + D(z_b, r; q) \leq C_1 < +\infty \quad (3.7)$$

for all  $z_b$  and for all  $r$  satisfying conditions

$$z_b = (be_3, 0), \quad b \in \mathbb{R}, \quad |b| \leq \frac{1}{4}, \quad 0 < r < \frac{1}{4}. \quad (3.8)$$

A constant  $C_1$  depends only on the constant  $C$  in (3.1),  $\|v\|_{L_3(Q)}$ , and  $\|q\|_{L_{\frac{3}{2}}(Q)}$ .

PROOF By Lemma 3.3 and by Remark 3.4, we have two inequalities:

$$A(0, 3/4; v) + E(0, 3/4; v) \leq C_2 < +\infty, \quad (3.9)$$

$$|x'| |v_\varphi(x, t)| \leq C_2 \quad \text{for a.a. } z = (x, t) \in Q(1/2). \quad (3.10)$$

Constant  $C_2$  depends on the same arguments as constant  $C_1$ .

It follows from (3.5), that, for  $s_1 = \frac{7}{4}$  and  $l_1 = 10$ , inequality (3.10) takes the form

$$m_1 = \frac{58}{7}, \quad \mu_1 = \frac{1}{58}, \quad M_{s_1, l_1}(z_b, r; \widehat{v}) \leq cC_2^{10},$$

$$C(z_b, r; \widehat{v}) \leq cA^{\frac{1}{58}}(z_b, r; v)(C_2^{10})^{\frac{7}{58}}(E(z_b, r; v) + A(z_b, r; v))^{\frac{51}{58}} \quad (3.11)$$

provided conditions (3.8) hold.

To treat  $\bar{v}$  which is the other part of the velocity  $v$ , we chose numbers  $s_2 = 4$  and  $l_2 = \frac{12}{7}$ . Then, for the same reasons as above, we find

$$m_2 = \frac{10}{7}, \quad \mu_2 = \frac{3}{14}, \quad M_{s_2, l_2}(z_b, r; \bar{v}) \leq cC^{\frac{12}{7}},$$

$$C(z_b, r; \bar{v}) \leq cA^{\frac{3}{14}}(z_b, r; v)(C^{\frac{12}{7}})^{\frac{7}{10}}(E(z_b, r; v) + A(z_b, r; v))^{\frac{3}{10}} \quad (3.12)$$

for all  $z_b$  and  $r$  satisfying conditions (3.8).

Adding (3.11) and (3.12), we show

$$C(z_b, r; v) \leq c\left(C(z_b, r; \bar{v}) + C(z_b, r; \widehat{v})\right) \leq$$

$$\leq c\left(A^{\frac{1}{58}}(z_b, r; v)C_2^{\frac{35}{29}}(E(z_b, r; v) + A(z_b, r; v))^{\frac{51}{58}} + \right. \quad (3.13)$$

$$\left. + A^{\frac{3}{14}}(z_b, r; v)C^{\frac{6}{5}}(E(z_b, r; v) + A(z_b, r; v))^{\frac{3}{10}}\right)$$

for the same  $z_b$  and  $r$  as above.

Applying Young's inequality in (3.13), we arrive at the important estimate

$$C(z_b, r; v) \leq \varepsilon(E(z_b, r; v) + A(z_b, r; v)) + f_1(\varepsilon, C, C_2), \quad (3.14)$$

provided conditions (3.8) hold. In (3.14), the positive number  $\varepsilon$  is a parameter to pick up later. The rest of the proof is routine. In addition to (3.14), we consider the local energy inequality

$$E(z_b, r/2; v) + A(z_b, r/2; v) \leq c\left(C^{\frac{2}{3}}(z_b, r; v) + C(z_b, r; v) + D(z_b, r; q)\right) \quad (3.15)$$

and the decay estimate for the pressure field

$$D(z_b, \varrho; q) \leq c\left[\frac{\varrho}{r}D(z_b, r; q) + \left(\frac{r}{\varrho}\right)^2 C(z_b, r; v)\right]. \quad (3.16)$$

Here,  $z_b$  and  $r$  satisfy conditions (3.8) and  $0 < \varrho \leq r$ . If we let

$$\mathcal{E}(r) = E(z_b, r; v) + A(z_b, r; v) + D(z_b, r; q),$$

then, for a fixed small positive number  $\vartheta$ , one can derive from (3.15) and (3.16) the following estimate

$$\begin{aligned} \mathcal{E}(\vartheta r) &\leq c \left( C^{\frac{2}{3}}(z_b, 2\vartheta r; v) + C(z_b, 2\vartheta r; v) + D(z_b, 2\vartheta r; q) + \right. \\ &\quad \left. + \vartheta D(z_b, r; q) + \frac{1}{\vartheta^2} C(z_b, r; v) \right) \leq \\ &\leq c \left[ \vartheta D(z_b, r; q) + \frac{1}{\vartheta^2} C(z_b, r; v) + \frac{1}{\vartheta^{\frac{4}{3}}} C^{\frac{2}{3}}(z_b, r; v) \right]. \end{aligned}$$

Now, the last two terms on the right hand side of the latter inequality can be majorized with the help of (3.14). As a result, we have

$$\mathcal{E}(\vartheta r) \leq c \left( \vartheta + \frac{\varepsilon}{\vartheta^2} \right) \mathcal{E}(r) + f_2(\varepsilon, \vartheta, C, C_2).$$

We first chose  $\vartheta$  so that  $c\vartheta < \frac{1}{4}$ , pick up  $\varepsilon$  to provide the inequality  $\frac{c\varepsilon}{\vartheta^2} < \frac{1}{4}$ , and then we find

$$\mathcal{E}(\vartheta r) \leq \frac{1}{2} \mathcal{E}(r) + f_3(C, C_2).$$

The latter inequality can be easily iterated. After simple calculations, we derive the relation

$$\begin{aligned} E(z_b, r; v) + A(z_b, r; v) + D(z_b, r; q) &\leq c \left( A(0, 1/2; v) + E(0, 1/2; v) + \right. \\ &\quad \left. + D(0, 1/2; q) + f_3(C, C_2) \right) \end{aligned}$$

with  $z_b$  and  $r$  satisfying conditions (3.8). Lemma 3.5 is proved.

To prove Theorem 3.2, we need an analogue of Lemma 3.5. Here, it is.

**Lemma 3.6.** *Under assumptions of Theorem 3.2, estimate (3.7) is valid as well with constant  $C_1$  depending only on the constant  $C$  in (3.3),  $\|v\|_{L_3(Q)}$ , and  $\|q\|_{L_{\frac{3}{2}}(Q)}$ .*

Lemma 3.6 is proved in the same way as Lemma 3.5 and even easier because main inequality (3.14) can be established with the help of the case  $s = s_1, l = l_1$  only.

As it follows from conditions of Theorem 3.2 and the statement of Lemma 3.3, the module of the velocity field grows not faster than  $C/|x'|$  as  $|x'| \rightarrow 0$ . Moreover, the corresponding estimate is uniform in time. However, it turns out to be true under conditions of Theorem 3.1 as well. More precisely, we have the following.

**Proposition 3.7.** *Assume that all conditions of Theorem 3.1 hold. Then*

$$|v(x, t)| \leq \frac{C_1}{|x'|} \quad (3.17)$$

for all  $z = (x, t) \in Q(1/8)$ . A constant  $C_1$  depends only on the constant  $C$  in (3.1),  $\|v\|_{L_3(Q)}$ , and  $\|q\|_{L_{\frac{3}{2}}(Q)}$ .

PROOF In view of (3.5), we can argue essentially as in [26].

Let us fix a point  $x_0 \in \mathcal{C}(1/8)$  and put  $r_0 = |x'_0|, b_0 = x_{03}$ . So, we have  $r_0 < \frac{1}{8}$  and  $|b_0| < \frac{1}{8}$ . Further, we introduce the following cylinders:

$$\mathcal{P}_{r_0}^1 = \{r_0 < |x'| < 2r_0, |x_3| < r_0\}, \quad \mathcal{P}_{r_0}^2 = \{r_0/4 < |x'| < 3r_0, |x_3| < 2r_0\}.$$

$$\mathcal{P}_{r_0}^1(b_0) = \mathcal{P}_{r_0}^1 + b_0 e_3, \quad \mathcal{P}_{r_0}^2(b_0) = \mathcal{P}_{r_0}^2 + b_0 e_3,$$

$$Q_{r_0}^1(b_0) = \mathcal{P}_{r_0}^1(b_0) \times ] - r_0^2, 0[, \quad Q_{r_0}^2(b_0) = \mathcal{P}_{r_0}^2(b_0) \times ] - (2r_0)^2, 0[.$$

Now, let us scale our functions so that

$$x = r_0 y + b_0 e_3, \quad t = r_0^2 s, \quad u(y, s) = r_0 v(x, t), \quad p(y, s) = r_0^2 q(x, t).$$

As it was shown in [26], there exists a continuous nondecreasing function  $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+, \mathbb{R}_+ = \{s > 0\}$ , such that

$$\begin{aligned} \sup_{(y,s) \in Q_1^1(0)} |u(y, s)| + |\nabla u(y, s)| &\leq \Phi \left( \sup_{-2^2 < s < 0} \int_{\mathcal{P}_1^2(0)} |u(y, s)|^2 dy \right. \\ &+ \int_{Q_1^2(0)} |\nabla u|^2 dy ds + \int_{Q_1^2(0)} |u|^3 dy ds + \int_{Q_1^2(0)} |p|^{\frac{3}{2}} dy ds \Big). \end{aligned} \quad (3.18)$$

After making inverse scaling in (3.18), we find

$$\begin{aligned} \sup_{z \in Q_{r_0}^1(b_0)} r_0 |u(x, t)| + r_0^2 |\nabla u(x, t)| &\leq \Phi \left( cA(z_{b_0}, 3r_0; v) + cE(z_{b_0}, 3r_0; v) + \right. \\ &\left. + cC(z_{b_0}, 3r_0; v) + cD(z_{b_0}, 3r_0; q) \right) \leq \Phi \left( 4cC_1 \right). \end{aligned}$$

It remains to apply Lemma 3.5 and complete the proof of the proposition. Proposition 3.7 is proved.

## 4 Proof of Theorems 3.1 and 3.2

Using Lemmata 3.3, 3.5, 3.6, Remark 3.4, Proposition 3.7 and scaling arguments, we may assume (without loss of generality) that our solution  $v$  and  $q$  have the following properties:

$$\sup_{0 < r \leq 1} \left( A(0, r; v) + E(0, r; v) + C(0, r; v) + D(0, r; q) \right) = A_1 < +\infty, \quad (4.1)$$

$$\operatorname{ess\,sup}_{z=(x,t) \in Q} |x'| |v(x, t)| = A_2 < +\infty. \quad (4.2)$$

We may also assume that the function  $v$  is Hölder continuous in the completion of the set  $\mathcal{C} \times ]-1, -a^2[$  for any  $0 < a < 1$ .

Introducing functions

$$H(t) = \sup_{x \in \mathcal{C}} |v(x, t)|, \quad h(t) = \sup_{-1 < \tau \leq t} H(\tau),$$

let us suppose that our statement is wrong, i.e.,  $z = 0$  is a singular point. Then there are sequences  $x_k \in \overline{\mathcal{C}}$  and  $-1 < t_k < 0$ , having the following properties:

$$h(t_k) = H(t_k) = M_k = |v(x_k, t_k)| \rightarrow +\infty \quad \text{as } k \rightarrow +\infty.$$

We scale our functions  $v$  and  $q$  so that scaled functions possess axial symmetry:

$$\begin{aligned} u^k(y, s) &= \lambda_k v(\lambda_k y', x_{3k} + \lambda_k y_3, t_k + \lambda_k^2 s), & \lambda_k &= \frac{1}{M_k}, \\ p^k(y, s) &= \lambda_k^2 q(\lambda_k y', x_{3k} + \lambda_k y_3, t_k + \lambda_k^2 s). \end{aligned}$$

These functions satisfy the Navier-Stokes equations in  $Q(M_k)$ . Moreover,

$$|u^k(y'_k, 0, 0)| = 1, \quad y'_k = M_k x'_k. \quad (4.3)$$

According to (4.2),

$$|y'_k| \leq A_2$$

for all  $k \in \mathbb{N}$ . Thus, without loss of generality, we may assume that

$$y'_k \rightarrow y'_* \quad \text{as } k \rightarrow +\infty. \quad (4.4)$$

Now, let us see what happens as  $k \rightarrow +\infty$ . By the identity

$$\sup_{e=(y,s) \in \mathcal{C}(M_k)} |u^k(e)| = 1 \quad (4.5)$$

and by (4.1), we can select subsequences (still denote as the entire sequence) such that

$$u^k \rightharpoonup^* u \quad \text{in } L_\infty(Q(a)), \quad (4.6)$$

and

$$p^k \rightharpoonup p \quad \text{in } L_{\frac{3}{2}}(Q(a)) \quad (4.7)$$

for any  $a > 0$ . Functions  $u$  and  $p$  are defined on  $Q_- = \mathbb{R}^3 \times ]-\infty, 0[$ . Obviously, they possess the following properties:

$$\text{ess sup}_{e \in Q_-} |u(e)| \leq 1, \quad (4.8)$$

$$\sup_{0 < r < +\infty} \left( A(0, r; u) + E(0, r; u) + C(0, r; u) + D(0, r; p) \right) \leq A_1, \quad (4.9)$$

$$\text{ess sup}_{e=(y,s) \in Q_-} |y'| |u(y, s)| \leq A_2. \quad (4.10)$$

Now, our aim is to show that  $u$  and  $p$  satisfy the Navier-Stokes equations  $Q_-$  and  $u$  is smooth enough to obey the identity

$$|u(y'_*, 0, 0)| = 1. \quad (4.11)$$

To this end, we fix an arbitrary positive number  $a > 0$  and consider numbers  $k$  so big that  $a < M_k/4$ . We know that  $u^k$  satisfies the nonhomogeneous heat equation of the form

$$\partial_t u^k - \Delta u^k = -\text{div } F^k \quad \text{in } Q(4a),$$

where  $F^k = u^k \otimes u^k + p^k \mathbb{I}$  and

$$\|F^k\|_{\frac{3}{2}, Q(4a)} \leq c_1(a) < \infty.$$

This implies the following fact, see [13],

$$\|\nabla u^k\|_{\frac{3}{2}, Q(3a)} \leq c_2(a) < \infty.$$

Now, we can interpret the pair  $u^k$  and  $p^k$  as a solution to the nonhomogeneous Stokes system

$$\partial_t u^k - \Delta u^k + \nabla p^k = f^k, \quad \operatorname{div} u^k = 0 \quad \text{in } Q(3a), \quad (4.12)$$

where  $f^k = -u^k \cdot \nabla u^k$  is the right hand side having the property

$$\|f^k\|_{\frac{3}{2}, Q(3a)} \leq c_2(a).$$

Then, according to the local regularity theory for the Stokes system, see [23], we can state that

$$\|\partial_t u^k\|_{\frac{3}{2}, Q(2a)} + \|\nabla^2 u^k\|_{\frac{3}{2}, Q(2a)} + \|\nabla k^k\|_{\frac{3}{2}, Q(2a)} \leq c_3(a).$$

The latter, together with the embedding theorem, implies

$$\|\nabla u^k\|_{3, \frac{3}{2}, (Q(2a))} + \|p^k\|_{3, \frac{3}{2}, Q(2a)} \leq c_4(a).$$

In turn, this improves integrability of the right hand side in (4.12)

$$\|f^k\|_{3, \frac{3}{2}, Q(2a)} \leq c_4(a).$$

Therefore, by the local regularity theory,

$$\|\partial_t u^k\|_{3, \frac{3}{2}, Q(2a)} + \|\nabla^2 u^k\|_{3, \frac{3}{2}, Q(2a)} + \|\nabla k^k\|_{3, \frac{3}{2}, Q(2)} \leq c_5(a).$$

Applying the imbedding theorem once more, we find

$$\|\nabla u^k\|_{6, \frac{3}{2}, Q(2a)} + \|p^k\|_{6, \frac{3}{2}, Q(2a)} \leq c_6(a).$$

The local regularity theory leads then to the estimate

$$\|\partial_t u^k\|_{6, \frac{3}{2}, Q(a)} + \|\nabla^2 u^k\|_{6, \frac{3}{2}, Q(a)} + \|\nabla p^k\|_{6, \frac{3}{2}, Q(a)} \leq c_7(a).$$

By the embedding theorem, sequence  $u^k$  is uniformly bounded in the parabolic Hölder space  $C^{\frac{1}{2}}(\overline{Q}(a/2))$ . Hence, without loss of generality, one may assume that

$$u^k \rightarrow u \quad \text{in } C^{\frac{1}{4}}(\overline{Q}(a/2)).$$

This means that the pair  $u$  and  $p$  obeys the Navier-Stokes system and (4.11) holds. So, the function  $u$  is the so-called bounded ancient solution to the Navier-Stokes system which is, in addition, axially symmetric and satisfies the decay estimate (4.10). As it was shown in [10], such a solution must be identically zero. But this contradicts (4.11). Theorems (3.1) and (3.2) are proved.

## 5 Appendix I: Proof of Theorem 2.4

In what follows, we need a few known regularity results.

**Lemma 5.1.** *Assume that functions  $f \in L_m(B(2))$  and  $q \in L_m(B(2))$  satisfy the equation*

$$\Delta q = -\operatorname{div} f \quad \text{in } B(2).$$

Then

$$\int_{B(1)} |\nabla q|^m dx \leq c(m, n) \left( \int_{B(2)} |f|^m dx + \int_{B(2)} |q - [q]_{B(2)}|^m dx \right).$$

**Lemma 5.2.** *Assume that functions  $f \in L_m(Q(2))$  and  $u \in W_m^{1,0}(Q(2))$  satisfy the equation*

$$\partial_t u - \Delta u = f \quad \text{in } Q(2).$$

Then  $u \in W_m^{2,1}(Q(1))$  and the following estimate is valid:

$$\|\partial_t u\|_{m,Q(1)} + \|\nabla^2 u\|_{m,Q(1)} \leq c(m, n) \left[ \|f\|_{m,Q(2)} + \|u\|_{W_m^{1,0}(Q(2))} \right].$$

Proof of Lemmata 5.1 and 5.2 can be found, for example, in [14] and [13].

**PROOF OF THEOREM 2.4: STEP 1. ENERGY ESTIMATE.** Take an arbitrary number  $t_0 < 0$  and fix it. Let  $k_\varepsilon(z)$  be a standard smoothing kernel and let

$$F^\varepsilon(z) = \int_{Q_-} k_\varepsilon(z - z') F(z') dz', \quad F = u \otimes u,$$

$$u^\varepsilon(z) = \int_{Q_-} k_\varepsilon(z - z')u(z')dz'.$$

Assume that  $w \in \mathring{C}_0^\infty(Q_-^{t_0})$ , where  $Q_-^{t_0} = \mathbb{R}^n \times ] - \infty, t_0[$ . Obviously,  $w^\varepsilon \in \mathring{C}_0^\infty(Q_-)$  for sufficiently small  $\varepsilon$ . Using known properties of smoothing kernel and Definition 2.3, we find

$$\int_{Q_-} w \cdot (\partial_t u^\varepsilon + \operatorname{div} F^\varepsilon - \Delta u^\varepsilon) dz = 0, \quad \forall w \in \mathring{C}_0^\infty(Q_-^{t_0}).$$

There exists a smooth function  $p_\varepsilon$  with the following property

$$\partial_t u^\varepsilon + \operatorname{div} F^\varepsilon - \Delta u^\varepsilon = -\nabla p_\varepsilon, \quad \operatorname{div} u^\varepsilon = 0 \quad (5.1)$$

in  $Q_-^{t_0}$ . Splitting pressure  $p_\varepsilon$  into two parts

$$p_\varepsilon = p_{F^\varepsilon} + \tilde{p}_\varepsilon. \quad (5.2)$$

and observing that the function  $\nabla p_{F^\varepsilon}$  is bounded in  $Q_-^{t_0}$ , one can conclude that, by (5.1) and (5.2),

$$\Delta \tilde{p}_\varepsilon = 0 \quad \text{in } Q_-^{t_0}, \quad \nabla \tilde{p}_\varepsilon \in L_\infty(Q_-^{t_0}; \mathbb{R}^n).$$

According to Liouville's theorem for harmonic functions,

$$\nabla \tilde{p}_\varepsilon(x, t) = a_\varepsilon(t), \quad x \in \mathbb{R}^n, \quad -\infty < t \leq t_0.$$

So, we have

$$\partial_t u^\varepsilon + \operatorname{div} F^\varepsilon - \Delta u^\varepsilon = -\nabla p_{F^\varepsilon} - a_\varepsilon, \quad \operatorname{div} u^\varepsilon = 0 \quad (5.3)$$

in  $Q_-^{t_0}$ .

Now, let us introduce new auxiliary functions

$$b_{\varepsilon t_0}(t) = \int_{t_0-1}^t a_\varepsilon(\tau) d\tau, \quad t_0 - 1 \leq t \leq t_0,$$

$$v_\varepsilon(x, t) = u^\varepsilon(x, t) + b_{\varepsilon t_0}(t), \quad z = (x, t) \in Q_-^{t_0}.$$

Using them, one may reduce system (5.3) to the form

$$\partial_t v_\varepsilon - \Delta v_\varepsilon = -\operatorname{div} F^\varepsilon - \nabla p_{F^\varepsilon}, \quad \operatorname{div} v_\varepsilon = 0 \quad (5.4)$$

in  $Q_-^{t_0}$ .

Let  $\varphi_{x_0}(x) = \varphi(x - x_0)$  for a fixed cut-off function  $\varphi$  satisfying the conditions

$$0 \leq \varphi \leq 1, \quad \varphi \equiv 1 \quad \text{in} \quad B(1), \quad \operatorname{supp} \varphi \subset B(2).$$

To derive the energy identity, let us multiply (5.4) by  $\varphi_{x_0}^2 v_\varepsilon$  and integrate the product by parts. As a result, we have

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^n} \varphi_{x_0}^2(x) |v_\varepsilon(x, t)|^2 dx + 2 \int_{t_0-1}^t \int_{\mathbb{R}^n} \varphi_{x_0}^2 |\nabla v_\varepsilon|^2 dx dt' = \\ &= \int_{\mathbb{R}^n} \varphi_{x_0}^2(x) |v_\varepsilon(x, t_0 - 1)|^2 dx + \int_{t_0-1}^t \int_{\mathbb{R}^n} \Delta \varphi_{x_0}^2 |v_\varepsilon|^2 dx dt' + \\ &\quad + \int_{t_0-1}^t \int_{\mathbb{R}^n} (p_{F^\varepsilon} - [p_{F^\varepsilon}]_{B(x_0, 2)}) v_\varepsilon \cdot \nabla \varphi_{x_0}^2 dx dt' + \\ &\quad + \int_{t_0-1}^t \int_{\mathbb{R}^n} (F^\varepsilon - [F^\varepsilon]_{B(x_0, 2)}) : \nabla (\varphi_{x_0}^2 v_\varepsilon) dx dt'. \end{aligned}$$

Introducing

$$\alpha_\varepsilon(t) = \sup_{x_0 \in \mathbb{R}^n} \int_{B(x_0, 1)} |v_\varepsilon(x, t)|^2 dx$$

and taking into account that  $v_\varepsilon(\cdot, t_0 - 1) = u^\varepsilon(\cdot, t_0 - 1)$  and  $|u^\varepsilon(\cdot, t_0 - 1)| \leq 1$ , we can estimate the right hand side of the energy identity in the following way

$$\begin{aligned} I(t) &\leq c(n) + c(n) \int_{t_0-1}^t \alpha_\varepsilon(t') dt' + \\ &+ c(n) \left( \int_{t_0-1}^{t_0} \int_{B(x_0, 2)} |p_{F^\varepsilon} - [p_{F^\varepsilon}]_{B(x_0, 2)}|^2 dx dt \right)^{\frac{1}{2}} \left( \int_{t_0-1}^t \alpha_\varepsilon(t') dt' \right)^{\frac{1}{2}} + \end{aligned}$$

$$\begin{aligned}
& + c(n) \left( \int_{t_0-1}^{t_0} \int_{B(x_0,2)} |F^\varepsilon - [F^\varepsilon]_{B(x_0,2)}|^2 dx dt \right)^{\frac{1}{2}} \left( \int_{t_0-1}^t \int_{\mathbb{R}^n} \varphi_{x_0}^2 |\nabla v_\varepsilon|^2 dx dt' + \right. \\
& \quad \left. + \int_{t_0-1}^t \alpha_\varepsilon(t') dt' \right)^{\frac{1}{2}}, \quad t_0 - 1 \leq t \leq t_0.
\end{aligned} \tag{5.5}$$

Next, since  $|F^\varepsilon| \leq c(n)$ , we find two estimates

$$\int_{t_0-1}^{t_0} \int_{B(x_0,2)} |F^\varepsilon - [F^\varepsilon]_{B(x_0,2)}|^2 dx dt \leq c(n)$$

and

$$\begin{aligned}
\int_{t_0-1}^{t_0} \int_{B(x_0,2)} |p_{F^\varepsilon} - [p_{F^\varepsilon}]_{B(x_0,2)}|^2 dx dt & \leq c(n) \|p_{F^\varepsilon}\|_{L^\infty(-\infty, t_0; BMO(\mathbb{R}^n))}^2 \\
& \leq c(n) \|F^\varepsilon\|_{L^\infty(Q_-^{t_0})}^2 \leq c(n).
\end{aligned}$$

The latter estimates, together with (5.5), imply the inequalities

$$\alpha_\varepsilon(t) \leq c(n) \left( 1 + \int_{t_0-1}^t \alpha_\varepsilon(t') dt' \right), \quad t_0 - 1 \leq t \leq t_0$$

and

$$\sup_{x_0 \in \mathbb{R}^n} \int_{t_0-1}^{t_0} \int_{B(x_0,1)} |\nabla v_\varepsilon|^2 dx dt \leq c(n) \left( 1 + \int_{t_0-1}^{t_0} \alpha_\varepsilon(t) dt \right).$$

Applying known arguments, we can conclude

$$\sup_{t_0-1 \leq t \leq t_0} \alpha_\varepsilon(t) + \sup_{x_0 \in \mathbb{R}^n} \int_{t_0-1}^{t_0} \int_{B(x_0,1)} |\nabla u^\varepsilon|^2 dx dt \leq c(n). \tag{5.6}$$

It should be emphasized that the right hand size in (5.6) is independent of  $t_0$ . In particular, estimate (5.6) allows to show

$$\sup_{t_0-1 \leq t \leq t_0} b_{\varepsilon t_0}(t) \leq c(n).$$

Now, let us see what happens if  $\varepsilon \rightarrow 0$ . Selecting a subsequence if necessary and taking the limit as  $\varepsilon \rightarrow 0$ , we state that:

$$b_{\varepsilon t_0} \xrightarrow{*} b_{t_0} \quad \text{in} \quad L_\infty(t_0 - 1, t_0; \mathbb{R}^n);$$

the estimate

$$\|b_{t_0}\|_{L_\infty(t_0-1, t_0)} + \sup_{x_0 \in \mathbb{R}^n} \int_{t_0-1}^{t_0} \int_{B(x_0, 1)} |\nabla u|^2 dx dt \leq c(n) < +\infty \quad (5.7)$$

is valid for all  $t_0 < 0$ ; the system

$$\partial_t u^{t_0} + \operatorname{div} u \otimes u - \Delta u = -\nabla p_{u \otimes u}, \quad \operatorname{div} u = 0$$

holds in  $Q^{t_0}$  in the sense of distributions.

The case  $t_0 = 0$  can be treated by passing to the limit as  $t_0 \rightarrow 0$ .

STEP 2, BOOTSTRAP ARGUMENTS By (5.7),

$$f = \operatorname{div} F = u \cdot \nabla u \in \mathcal{L}_2(Q_-; \mathbb{R}^n).$$

Then Lemma 5.1, together with shifts, shows that

$$\nabla p_{u \otimes u} \in \mathcal{L}_2(Q_-; \mathbb{R}^n).$$

Next, obviously, the function  $u^{t_0}$  satisfies the system of equations

$$\partial_t u^{t_0} - \Delta u^{t_0} = -u \cdot \nabla u - \nabla p_{u \otimes u} \in \mathcal{L}_2(Q_-; \mathbb{R}^n),$$

which allows us to apply Lemma 5.2 and conclude that

$$u^{t_0} \in W_2^{2,1}(Q(z_0, \tau_2); \mathbb{R}^n), \quad 1/2 < \tau_2 < \tau_1 = 1.$$

Moreover, the estimate

$$\|u^{t_0}\|_{W_2^{2,1}(Q(z_0, \tau_2))} \leq c(n, \tau_2)$$

holds for any  $z_0 = (x_0, t_0)$ , where  $x_0 \in \mathbb{R}^n$  and  $t_0 \leq 0$ . Applying the parabolic embedding theorem, see [13], we can state that

$$\nabla u^{t_0} = \nabla u \in W_{m_2}^{1,0}(Q(z_0, \tau_2); \mathbb{R}^n),$$

where

$$\frac{1}{m_2} = \frac{1}{m_1} - \frac{1}{n+2}, \quad m_1 = 2.$$

By Lemma 5.1, by shifts, and by scaling,

$$\int_{B(x_0, \tau'_3)} |\nabla p_{u \otimes u}(\cdot, t)|^{m_2} dx \leq c(n, \tau_2, \tau'_3) \left[ \int_{B(x_0, \tau'_3)} |\nabla u(\cdot, t)|^{m_2} dx + 1 \right]$$

for  $1/2 < \tau'_3 < \tau_2$ . In turn, Lemma 5.2 provides two statements:

$$u^{t_0} \in W_{m_2}^{2,1}(Q(z_0, \tau_3); \mathbb{R}^n), \quad 1/2 < \tau_3 < \tau'_3$$

and

$$\|u^{t_0}\|_{W_{m_2}^{2,1}(Q(z_0, \tau_3))} \leq c(n, \tau_3, \tau'_3).$$

Then, again, by the embedding theorem, we find

$$\nabla u^{t_0} = \nabla u \in W_{m_3}^{1,0}(Q(z_0, \tau_3); \mathbb{R}^n)$$

with

$$\frac{1}{m_3} = \frac{1}{m_2} - \frac{1}{n+2}.$$

Now, let us take an arbitrary large number  $m > 2$  and fix it. Find  $\alpha$  as an unique solution to the equation

$$\frac{1}{m} = \frac{1}{2} - \frac{\alpha}{n+2}.$$

Next, for  $k_0 = [\alpha] + 1$ , where  $[\alpha]$  is the entire part of the number  $\alpha$ , determine the number  $m_{k_0+1}$  satisfying the identity

$$\frac{1}{m_{k_0+1}} = \frac{1}{2} - \frac{k_0}{n+2}.$$

Obviously,  $m_{k_0+1} > m$ . Setting

$$\tau_{k+1} = \tau_k - \frac{1}{4} \frac{1}{2^k}, \quad \tau_1 = 1, \quad k = 1, 2, \dots,$$

and repeating our previous arguments  $k_0$  times, we conclude that:

$$u^{t_0} \in W_{m_{k_0+1}}^{2,1}(Q(z_0, \tau_{k_0+1}); \mathbb{R}^n)$$

and

$$\|u^{t_0}\|_{W_{m_{k_0+1}}^{2,1}(Q(z_0, \tau_{k_0+1}))} \leq c(n, m).$$

Since  $\tau_k > 1/2$  for any natural numbers  $k$ , we complete the proof of Theorem 2.4. Theorem 2.4 is proved.

We can exclude the pressure field completely by considering the equations for vorticity  $\omega = \nabla \wedge u$ . In dimensions three, differentiability properties of  $\omega$  are described by the following theorem.

**Lemma 5.3.** *Let  $u$  be an arbitrary bounded ancient solution. For any  $m > 1$ , we have*

$$\omega = \nabla \wedge u \in \mathcal{W}_m^{2,1}(Q_-; \mathbb{R}^3)$$

and

$$\partial_t \omega + u \cdot \nabla \omega - \Delta \omega = \omega \cdot \nabla u \quad \text{a.e. in } Q_-.$$

**Remark 5.4.** *We could continue investigations of regularity for solutions to the vorticity equations further and it would be a good exercise. However, regularity results stated in Theorem 5.3 are sufficient for our purposes.*

**Remark 5.5.** *Functions  $\omega$  and  $\nabla \omega$  are Hölder continuous in  $Q_-$  and their norms in Hölder spaces are uniformly bounded there, see [13].*

**PROOF OF LEMMA 5.3** Let us consider the case  $n = 3$ . The case  $n = 2$  is in fact easier. So, we have

$$\partial_t \omega - \Delta \omega = \omega \cdot \nabla u - u \cdot \nabla \omega \equiv f.$$

Take an arbitrary number  $m > 2$  and fix it. By Theorem 2.4,

$$|f| \leq c(n)(|\nabla^2 u| + |\nabla u|^2) \in L_m(Q(z_0, 2))$$

and the norm of  $f$  in  $L_m(Q(z_0, 2))$  is dominated by a constant depending only on  $m$ . It remains to apply Lemma 5.2 and complete the proof of Lemma 5.3. Lemma 5.3 is proved.

## 6 Appendix II: Proof of Lemma 3.3

According to the local regularity theory of the Navier-Stokes equations, see, for instance, [23], one can easily show that the pair  $v$  and  $q$  has the following differentiability properties:

$$v \in W_{\frac{3}{2}}^{2,1}(\mathcal{C}(a) \times ]-a^2, -b^2[), \quad q \in W_{\frac{3}{2}}^{1,0}(\mathcal{C}(a) \times ]-a^2, -b^2[) \quad (6.1)$$

and

$$v \text{ is Hölder continuous in the completion of } \mathcal{C}(a) \times ] - a^2, -b^2[ \quad (6.2)$$

for any  $0 < b \leq a < 1$ .

Now, we fix a number  $m \geq 2$ , multiply the equation for the velocity component  $v_\varphi$  by  $ru|u|^{m-2}$ , where  $u = rv_\varphi$ , and integrate the product by parts. In view of (6.1) and (6.2), we find the following identity for  $\omega = |u|^m$

$$\begin{aligned} & \int_{\mathcal{C}} \psi^2(x, t_*) |\omega(x, t_*)|^2 dx + \frac{2(m-1)}{m} \int_{-1}^{t_*} \int_{\mathcal{C}} \psi^2 |\nabla \omega(x, t)|^2 dx dt \\ &= \int_{-1}^{t_*} \int_{\mathcal{C}} |\omega(x, t)|^2 \left( \partial_t \psi^2 + \bar{v} \cdot \nabla \psi^2 + \Delta_2 \psi^2 + \frac{3\psi_{,r}^2}{r} \right) dx dt. \end{aligned} \quad (6.3)$$

It is valid for all  $-1 < t_* < 0$  and for all cut-off functions vanishing in a neighborhood of the boundary of the space-time cylinder  $\mathcal{C} \times ] - 1, 1[$ . Here,  $\Delta_2 \psi^2 = \psi_{,rr}^2 + \psi_{,33}^2$ . So, (6.3) means that the energy norm of  $\psi\omega$  is finite, i.e.,

$$\begin{aligned} & |\psi\omega|_{2, \mathcal{C} \times ] - 1, t_*[}^2 \equiv \text{ess sup}_{t \in ] - 1, t_*[} \int_{\mathcal{C}} |\psi\omega(x, t)|^2 dx + \int_{-1}^{t_*} \int_{\mathcal{C}} |\nabla(\psi\omega)|^2 dx dt \\ & \leq c \int_{-1}^{t_*} \int_{\mathcal{C}} |\omega(x, t)|^2 \left( \partial_t \psi^2 + \bar{v} \cdot \nabla \psi^2 + \Delta_2 \psi^2 + \frac{3\psi_{,r}^2}{r} + |\nabla \psi|^2 \right) dx dt \end{aligned} \quad (6.4)$$

for any  $-1 < t_* < 0$ .

No, let us specify our cut-off function  $\psi$  setting  $\psi(x, t) = \Phi(x)\chi(t)$  and assuming that new smooth functions  $0 \leq \Phi \leq 1$  and  $0 \leq \chi \leq 1$  meet the following properties:

$$\text{supp } \Phi \in \mathcal{C}(r_1), \quad \Phi \equiv 1 \text{ in } \mathcal{C}(r),$$

$$|\nabla \Phi| \leq \frac{c}{r_1 - r}, \quad |\nabla^2 \Phi| \leq \frac{c}{(r_1 - r)^2}, \quad |\partial_t \chi| \leq \frac{c}{(r_1 - r)^2}.$$

Here, arbitrary fixed number  $r$  and  $r_1$  satisfy the condition

$$\frac{1}{2} < r < r_1 < \frac{3}{4}. \quad (6.5)$$

If we let  $\tilde{Q} = \mathcal{C}(r_1) \times ] - r_1^2, t_*[$ , then

$$|\psi\omega|_{2,\tilde{Q}}^2 \leq \frac{c}{(r_1 - r)^2} \int_{\tilde{Q}} |\omega|^2 dz + cI, \quad (6.6)$$

where

$$I = \frac{1}{r_1 - r} \int_{\tilde{Q}} |\psi\omega| |\omega| |\bar{v}| dz.$$

By Hölder's inequality,

$$I \leq \frac{1}{r_1 - r} \left( \int_{\tilde{Q}} |\bar{v}|^{\frac{10}{3}} dz \right)^{\frac{3}{10}} \left( \int_{\tilde{Q}} |\omega|^{\frac{5}{2}} dz \right)^{\frac{2}{5}} \left( \int_{\tilde{Q}} |\psi\omega|^{\frac{10}{3}} dz \right)^{\frac{3}{10}}.$$

The left hand side of (6.5) can be evaluated from below with the help of the well-known multiplicative inequality

$$\left( \int_{\tilde{Q}} |\psi\omega|^{\frac{10}{3}} dz \right)^{\frac{3}{10}} \leq c |\psi\omega|_{2,\tilde{Q}}. \quad (6.7)$$

Now, taking into account restriction (6.5), it is not so difficult to derive from (6.6) and (6.7) the following estimate

$$\left( \int_{\tilde{Q}} |\psi\omega|^{\frac{10}{3}} dz \right)^{\frac{3}{10}} \leq cM \frac{\sqrt{r_1}}{r_1 - r} \left( \int_{\tilde{Q}} |\omega|^{\frac{5}{2}} dz \right)^{\frac{2}{5}}. \quad (6.8)$$

Setting

$$m = m_k = \left(\frac{4}{3}\right)^k, \quad r_1 = r^{(k)} = \frac{1}{2} + \frac{1}{2^{k+1}}, \quad r = r^{(k+1)},$$

$$\psi = \psi_k, \quad \tilde{Q}_k = \mathcal{C}(r^{(k)}) \times ] - (r^{(k)})^2, t_*[, \quad k \in \mathbb{N}.$$

one can reduce (6.8) to the form

$$\left( \int_{\tilde{Q}_k} |\psi_k u|^{\frac{10m_k}{3}} dz \right)^{\frac{3}{10m_k}} \leq cM \frac{\sqrt{r^{(k)}}}{r^{(k)} - r^{(k+1)}} \left( \int_{\tilde{Q}_k} |u|^{\frac{5m_k}{2}} dz \right)^{\frac{2}{5m_k}} \quad (6.9)$$

The only difference with respect to the usual Moser's technique is that one should take the limit as  $t_* \rightarrow 0$  step-by-step. For example, for  $k = 1$ , the integral

$$\int_{Q(3/4)} |u|^{\frac{10}{3}} dz$$

is finite and therefore we can pass to the limit as  $t_* \rightarrow 0$  in (6.8). Then we may pass to the limit as  $t_* \rightarrow 0$  in (6.8) for  $k = 2$  and so on. Tending  $k \rightarrow +\infty$ , we complete the proof of Lemma 3.3 in more or less standard way. Lemma 3.3 is proved.

## References

- [1] Caffarelli, L., Kohn, R.-V., Nirenberg, L., Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.*, Vol. XXXV (1982), pp. 771–831.
- [2] Chae D., Lee, J., On the regularity of the axisymmetric solutions of the Navier-Stokes equations, *Math. Z.*, 239(2002), 645-671.
- [3] Chen, C.-C., Strain, R., Tsai, T.-P., Yau, H.-T., Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations, downloaded from arXiv e-prints server.
- [4] Chen, C.-C., Strain, R., Tsai, T.-P., Yau, H.-T., Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations II, preprint arXiv:0709.4230.
- [5] Escauriaza, L., Seregin, G., Šverák, V.,  $L_{3,\infty}$ -Solutions to the Navier-Stokes equations and backward uniqueness, *Uspekhi Matematicheskikh Nauk*, v. 58, 2(350), pp. 3–44. English translation in *Russian Mathematical Surveys*, 58(2003)2, pp. 211-250.
- [6] Frehse, J., Ruzichka, M., Existence of regular solutions to the stationary Navier-Stokes equations, *Math. Anal.* 302(1995), pp. 699-717.
- [7] Gidas, B., Spruck, J., A priori bounds for positive solutions of nonlinear elliptic equations, *Comm. Partial Differential Equations* 6 (1981), no. 8, 883–901.

- [8] Giga, Y., Kohn, R., Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.* 38 (1985), no. 3, 297–319.
- [9] Hamilton, R., The formation of singularities in the Ricci flow. *Surveys in differential geometry*, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.
- [10] Koch, G., Nadirashvili, N., Seregin, G., and Sverak V., Liouville theorems for Navier-Stokes equations and applications, downloaded from arXiv e-prints server.
- [11] Ladyzhenskaya, O. A., On unique solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations under the axial symmetry, *Zap. Nauchn. Sem. LOMI* 7(1968), 155-177.
- [12] Ladyzhenskaya, O. A., Seregin, G. A., On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations, *J. math. fluid mech.*, 1(1999), pp. 356-387.
- [13] Ladyzhenskaya, O. A., Solonnikov, V. A., Uralt'seva, N. N., *Linear and quasi-linear equations of parabolic type*, Moscow, 1967; English translation, American Math. Soc., Providence 1968.
- [14] Ladyzhenskaya, O. A., Uraltseva, N. N., *Linear and quasilinear equations of elliptic type*, "Nauka", Moscow, 1973.
- [15] Lemarie-Riesset, P. G., *Recent developments in the Navier-Stokes problem*, Chapman&Hall/CRC research notes in mathematics series, 431 pp.
- [16] Leonardi, S., Malek, Necas, J., & Pokorný, M., On axially symmetric flows in  $\mathbb{R}^3$ , *ZAA*, 18(1999), 639-649.
- [17] Lin, F.-H., A new proof of the Caffarelli-Kohn-Nirenberg theorem, *Comm. Pure Appl. Math.*, 51(1998), no.3, pp. 241–257.
- [18] Nečas, J., Ruzička, M., Šverák, V., On Leray's self-similar solutions of the Navier-Stokes equations, *Acta Math.*, 176(1996), pp. 283-294.
- [19] Neustupa, J., Pokorný, M., Axisymmetric flow of Navier-Stokes fluid in the whole space with non-zero angular velocity components, *Math. Bohemica*, 126(2001), 469-481.

- [20] Pokorný, M., A regularity criterion for the angular velocity component in the case of axisymmetric Navier-Stokes equations, 2001.
- [21] Poláčik, P., Quittner, P., Souplet, P., Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part I: Elliptic equations and systems. *Duke Math. J.* 139 (2007), 555-579.
- [22] Poláčik, P., Quittner, P., Souplet, P., Singularity and decay estimates in superlinear problems via Liouville-type theorems. Part II: Parabolic equations, *Indiana Univ. Math. J.* 56 (2007), 879-908.
- [23] Seregin, G., Local regularity theory of the Navier-Stokes equations, to appear in *Handbook of Mathematical Fluid Mechanics*, vol. 4.
- [24] Seregin, G., Estimates of suitable weak solutions to the Navier-Stokes equations in critical Morrey spaces, *Zapiski Nauchn. Seminar POMI*, 336(2006), pp 199-210.
- [25] Seregin, G., Zajaczkowski, W., A sufficient condition of local regularity for the Navier-Stokes equations, *Zapiski Nauchn. Seminar, POMI*, 336(2006), pp. 46-54.
- [26] Seregin, G., Zajaczkowski, W., A sufficient condition of regularity for axially symmetric solutions to the Navier-Stokes equations, *SIMA J. Math. Anal.*, (39)2007, pp. 669–685.
- [27] Serrin, J., Zhou, H., Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, *Acta Math.* 189 (2002), 79–142.
- [28] Stein, E., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.
- [29] Tsai, T.-P., On Lerays Self-Similar Solutions of the Navier-Stokes Equations Satisfying Local Energy Estimates, *Arch. Rational Mech. Anal.* 143 (1998) 29–51.
- [30] Ukhovskij, M. R., Yudovich, V. L., Axially symmetric motions of ideal and viscous fluids filling all space, *Prikl. Mat. Mech.* 32 (1968), 59-69.

- [31] Zajaczkowski, W. M., Global special regular solutions to the Navier-Stokes equations in axially symmetric domains under boundary slip conditions, *Diss. Math.*, 400(2005).
- [32] Zhang, Qi S., A strong regularity result for parabolic equations, *Commun. Math. Phys.* 244(2004), pp. 245-260.