MATH 523: Homework

March 26, 2011 (Due date: March 31, 2011)

A. Find the three-dimensional Fourier transform for the function $f = f(X)$, $X \in \mathbb{R}^3$, given by

$$f(X) = 1 \text{ if } |X| \leq R, \quad f = 0 \text{ otherwise.}$$

Hint: evaluate the integral

$$\iint_{|Y|\leq R} e^{-iX\cdot Y} dY,$$

using spherical coordinates (r, θ, φ) rotated so that the polar axis $\theta = 0$ points in the direction of $\vec{X}$. Then you have $X \cdot Y = |X| r \cos\theta$, and $dY = r^2 \sin\theta dr d\theta d\varphi$.

B. Use **A.** to evaluate the surface integral

$$\int_{|Y|=R} e^{-iX\cdot Y} d\sigma(Y).$$

C. Let $\mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n : |\omega = 1\}$ be the unit sphere centered at the origin. Prove that the function $u(X,t) = e^{i\sqrt{\lambda}t}\psi(x)$, $(X,t) \in \mathbb{R}^{n+1}$, with $\psi \in C_0^\infty(\mathbb{R}^n)$ given by

$$\psi(x) = \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}X\cdot\omega} d\sigma(\omega), \quad \lambda > 0, \quad X \in \mathbb{R}^n,$$

solves the wave equation $\Delta u - \partial_t^2 u = 0$ in $\mathbb{R}^{n+1}$.

Use **B.** to write explicit formula for the solution when $n = 3$.