MATH 523: Homework
March 26, 2011 (Due date: March 31, 2011)

A. Find the three-dimensional Fourier transform for the function \( f = f(X), \ X \in \mathbb{R}^3, \) given by
\[
f(X) = 1 \text{ if } |X| \leq R, \quad f = 0 \text{ otherwise.}
\]
Hint: evaluate the integral
\[
\int \int_{|Y| \leq R} e^{-iX \cdot Y} dY,
\]
using spherical coordinates \((r, \theta, \varphi)\) rotated so that the polar axis \(\theta = 0\) points in the direction of \(\vec{X}.\) Then you have \(X \cdot Y = |X| r \cos \theta,\) and \(dY = r^2 \sin \theta dr d\theta d\varphi.\)

B. Use A. to evaluate the surface integral
\[
\int_{|Y| = R} e^{-iX \cdot Y} d\sigma(Y).
\]

C. Let \(S^{n-1} = \{\omega \in \mathbb{R}^n : |\omega| = 1\}\) be the unit sphere centered at the origin. Prove that the function
\[
u(X, t) = e^{i \sqrt{\lambda} t} \psi(x), \ (X, t) \in \mathbb{R}^{n+1}, \text{ with } \psi \in C_0^\infty(\mathbb{R}^n)
\]
given by
\[
\psi(x) = \int_{S^{n-1}} e^{i \sqrt{\lambda} X \cdot \omega} d\sigma(\omega), \quad \lambda > 0, \quad X \in \mathbb{R}^n,
\]
solves the wave equation \(\Delta u - \partial_t^2 u = 0\) in \(\mathbb{R}^{n+1}.\)
Use B. to write explicit formula for the solution when \(n = 3.\)