Problem 1

Proof. 1. Choose $w \in \mathcal{A}$. Then (46) implies

$$0 = \int_U (-\Delta u - f)(u - w)\, dx.$$

An integration by parts yields

$$0 = \int_U Du \cdot D(u-w) - f(u-w)\, dx,$$

and there is no boundary term since $u - w = g - g \equiv 0$ on ∂U. Hence

$$\begin{aligned}\int_U |Du|^2 - uf\, dx &= \int_U Du \cdot Dw - wf\, dx \\ &\leq \int_U \frac{1}{2}|Du|^2\, dx + \int_U \frac{1}{2}|Dw|^2 - wf\, dx,\end{aligned}$$

where we employed the estimates

$$|Du \cdot Dw| \leq |Du||Dw| \leq \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2,$$

following from the Cauchy–Schwarz and Cauchy inequalities (§B.2). Rearranging, we conclude

(48) $$I[u] \leq I[w] \quad (w \in \mathcal{A}).$$

Since $u \in \mathcal{A}$, (47) follows from (48).

2. Now, conversely, suppose (47) holds. Fix any $v \in C_c^\infty(U)$ and write

$$i(\tau) := I[u + \tau v] \quad (\tau \in \mathbb{R}).$$

Since $u + \tau v \in \mathcal{A}$ for each τ, the scalar function $i(\cdot)$ has a minimum at zero, and thus

$$i'(0) = 0 \quad \left(' = \frac{d}{d\tau}\right),$$

provided this derivative exists. But

$$\begin{aligned}i(\tau) &= \int_U \frac{1}{2}|Du + \tau Dv|^2 - (u + \tau v) f\, dx \\ &= \int_U \frac{1}{2}|Du|^2 + \tau Du \cdot Dv + \frac{\tau^2}{2}|Dv|^2 - (u+\tau v) f\, dx.\end{aligned}$$

Consequently

$$0 = i'(0) = \int_U Du \cdot Dv - vf\, dx = \int_U (-\Delta u - f) v\, dx.$$

This identity is valid for each function $v \in C_c^\infty(U)$ and so $-\Delta u = f$ in U. □

Dirichlet's principle is an instance of the *calculus of variations* applied to Laplace's equation. See Chapter 8 for more.

We have already employed the maximum principle in §2.2.3 to show uniqueness, but now set forth a simple alternative proof. Assume U is open, bounded, and ∂U is C^1.

THEOREM 16 (Uniqueness). *There exists at most one solution $u \in C^2(\bar{U})$ of (46).*

Proof. Assume $\tilde{u}$ is another solution and set $w := u - \tilde{u}$. Then $\Delta w = 0$ in U, and so an integration by parts shows

$$0 = -\int_U w\Delta w \, dx = \int_U |Dw|^2 dx.$$

Thus $Dw \equiv 0$ in U, and, since $w = 0$ on ∂U, we deduce $w = u - \tilde{u} \equiv 0$ in U. □

b. Dirichlet's principle.

Next let us demonstrate that a solution of the boundary-value problem (46) for Poisson's equation can be characterized as the minimizer of an appropriate functional. For this, we define the *energy* functional

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - wf \, dx,$$

w belonging to the *admissible set*

$$\mathcal{A} := \{w \in C^2(\bar{U}) \mid w = g \text{ on } \partial U\}.$$

THEOREM 17 (Dirichlet's principle). *Assume $u \in C^2(\bar{U})$ solves (46). Then*

(47)
$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Conversely, if $u \in \mathcal{A}$ satisfies (47), then u solves the boundary-value problem (46).

In other words if $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that u minimizes the energy $I[\cdot]$.

7.1.1f

$$(f)\quad \frac{1}{\sqrt{2\pi}\,(-k^2+2\,\mathrm{i}\,k+2)} = \frac{-k^2-2\,\mathrm{i}\,k+2}{\sqrt{2\pi}\,(k^4+4)}\,.$$

7.1.3a,b

(a) By the Shift Theorem 7.4, $f(x) = \mathrm{i}\sqrt{\dfrac{\pi}{2}}\, e^{-\mathrm{i}\,a\,x}\,\operatorname{sign} x$.

(b) Using the Table, if $b > 0$, then $f(x) = \mathrm{i}\sqrt{2\pi}\, e^{b\,x}\,(\sigma(x) - 1)$, while if $b < 0$, then $f(x) = \mathrm{i}\sqrt{2\pi}\, e^{b\,x}\,\sigma(x)$. For $b = 0$, use part (a).

7.1.13

Use the change of variables $\widehat{x} = x - \xi$ in the integral:

$$\mathcal{F}[\,f(x-\xi)\,] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\,e^{-\mathrm{i}\,k\,x}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\widehat{x})\,e^{-\mathrm{i}\,k\,(\widehat{x}+\xi)}\,d\widehat{x}$$
$$= \frac{e^{-\mathrm{i}\,k\,\xi}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\widehat{x})\,e^{-\mathrm{i}\,k\,\widehat{x}}\,d\widehat{x} = e^{-\mathrm{i}\,k\,\xi}\,\widehat{f}(k).$$

To prove the second statement,

$$\mathcal{F}[\,e^{\mathrm{i}\,\kappa\,x}\,f(x)\,] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-\mathrm{i}\,(k-\kappa)\,x}\,dx = \widehat{f}(k-\kappa).$$

7.1.20 a): (i), (iii), and b)

. (a) (i) $\dfrac{2}{\pi(k^2+1)(l^2+1)}$, ★ (iii) $\dfrac{e^{-\mathrm{i}\,(\xi\,k+\eta\,l)}}{2\pi}$,

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7.2.2a

$$(a)\quad -\frac{\mathrm{i}}{k}\sqrt{\frac{2}{\pi}}\,e^{-k^2/4} + \sqrt{2\pi}\,\delta(k).$$

7.2.3d

★ (d) $-\dfrac{d^2}{dx^2}\left[\sqrt{2\pi}\,e^{-x}\sigma(x)\right] = \sqrt{2\pi}\left[-e^{-x}\sigma(x) + \delta(x) - \delta'(x)\right].$

7.2.12

(a) Indeed, applying the inverse Fourier transform:

$$f(x) \sim \int_{-\infty}^{\infty} \widehat{f}(k)\,e^{\,i\,k\,x}\,dk = \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \delta(k-n)\,e^{\,i\,k\,x}\,dk = \sum_{n=-\infty}^{\infty} c_n e^{\,i\,k\,x}$$

recovers the complex Fourier series for $f(x)$, proving the result.

(b) (*i*) $\frac{1}{2}\,i\,\delta(x+2) - \frac{1}{2}\,i\,\delta(x-2)$, (*iii*) $i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{(-1)^n}{n}\,\delta(k-n).$

7.3.4

. The Fourier transformed equation is $(k^2+4)\,\widehat{u}(k) = 1/\sqrt{2\pi}$, and hence a solution is $u(x) = \frac{1}{4}\,e^{-2\,|\,x\,|}.$