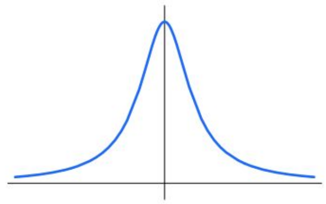
Homework 2 Solutions

2.2.17

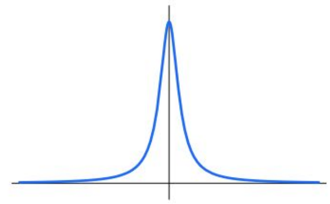
2.2.17. (a) $u(t,x) = \dfrac{1}{(x\,e^t)^2 + 1} = \dfrac{e^{-2t}}{x^2 + e^{-2t}}$.

(b)

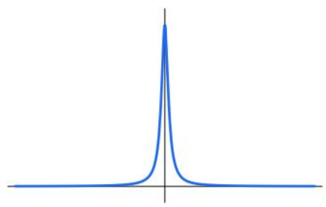
$t = 0$:



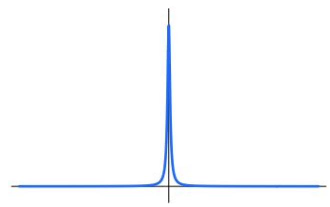
$t = 1$:



$t = 2$:



$t = 3$:



(c) The limit is discontinuous: $\displaystyle\lim_{t\to\infty} u(t,x) = \begin{cases} 1, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$

2.2.18

(a) $\displaystyle\lim_{t\to\infty} u(t,x) = \begin{cases} 0, & x < -1, \\ f(-1), & x = -1, \\ f(1), & x > -1. \end{cases}$ (b) $\displaystyle\lim_{t\to-\infty} u(t,x) = \begin{cases} f(-1), & x < 1, \\ f(1), & x = 1, \\ 0, & x > 1. \end{cases}$

2.2.26

(a) Suppose $x = x(t)$ solves $\dfrac{dx}{dt} = c(t,x)$. Then, by the chain rule,

$$\frac{d}{dt}\,u\big(t,x(t)\big) = \frac{\partial u}{\partial t}\big(t,x(t)\big) + \frac{\partial u}{\partial x}\big(t,x(t)\big)\frac{dx}{dt} = \frac{\partial u}{\partial t}\big(t,x(t)\big) + c\big(t,x(t)\big)\frac{\partial u}{\partial x}\big(t,x(t)\big) = 0,$$

since we are assuming that $u(t,x)$ is a solution to the transport equation for all (t,x). We conclude that $u\big(t,x(t)\big)$ is constant.

(b) Since $\xi(t,x) = k$ implicitly defines a solution $x(t)$ to the characteristic equation,

$$0 = \frac{d}{dt}\,\xi\big(t,x(t)\big) = \frac{\partial \xi}{\partial t}\big(t,x(t)\big) + \frac{\partial \xi}{\partial x}\big(t,x(t)\big)\frac{dx}{dt} = \frac{\partial \xi}{\partial t}(t,x) + c(t,x)\frac{\partial \xi}{\partial x}(t,x),$$

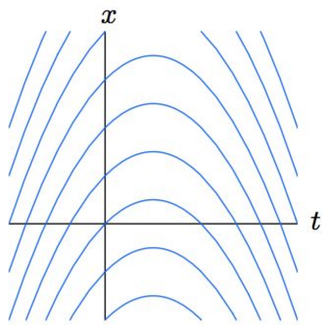
and hence $u = \xi(t,x)$ is a solution to the transport equation. Moreover, if $u(t,x) = f\big(\xi(t,x)\big)$, by the chain rule,

$$\frac{\partial u}{\partial t}(t,x) + c(t,x)\frac{\partial u}{\partial x}(t,x) = f'\big(\xi(t,x)\big)\left(\frac{\partial \xi}{\partial t}(t,x) + c(t,x)\frac{\partial \xi}{\partial x}(t,x)\right) = 0$$

according to the previous computation.

2.2.29

2.29. (a) Solving the characteristic equation $\dfrac{dx}{dt} = 1 - 2t$ produces the characteristic curves $x = t - t^2 + k$, where k is an arbitrary constant.



(b) The general solution is $u(t,x) = v(x - t + t^2)$, where $v(\xi)$ is an arbitrary C^1 function of the characteristic variable $\xi = x - t + t^2$.

(c) $u(t,x) = \dfrac{1}{1 + (x - t + t^2)^2}$.

(d) The solution is a hump of fixed shape that, as t increases, first moves to the right, slowing down and stopping at $t = \frac{1}{2}$, and then moving back to the left, at an ever accelerating speed. As $t \to -\infty$, the hump moves back to the left, accelerating.

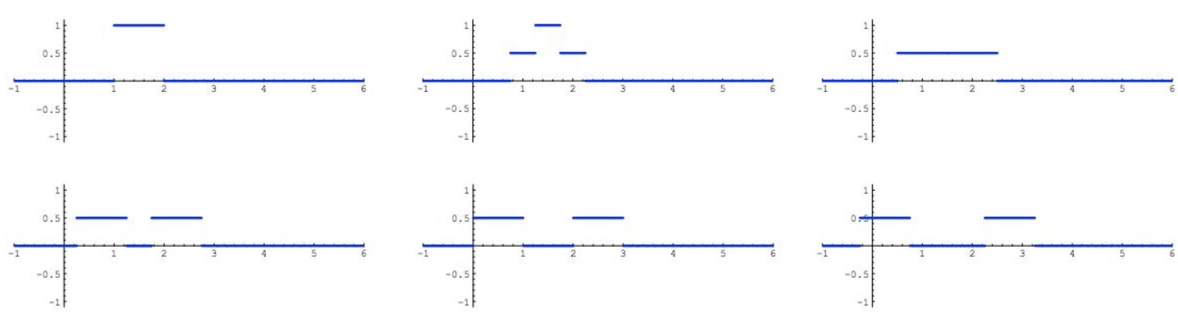
2.4.2

(a) The initial displacement splits into two half sized replicas, moving off to the right and to the left with unit speed.

For $t < \frac{1}{2}$, we have $u(t,x) = \begin{cases} 1, & 1 + t < x < 2 - t, \\ \frac{1}{2}, & 1 - t < x < 1 + t \quad \text{or} \quad 2 - t < x < 2 + t, \\ 0, & \text{otherwise,} \end{cases}$

For $t \geq \frac{1}{2}$, we have $u(t,x) = \begin{cases} \frac{1}{2}, & 1 - t < x < 2 - t \quad \text{or} \quad 1 + t < x < 2 + t, \\ 0, & \text{otherwise,} \end{cases}$

(b) Plotted at times $t = 0, .25, .5, .75, 1., 1.25$:



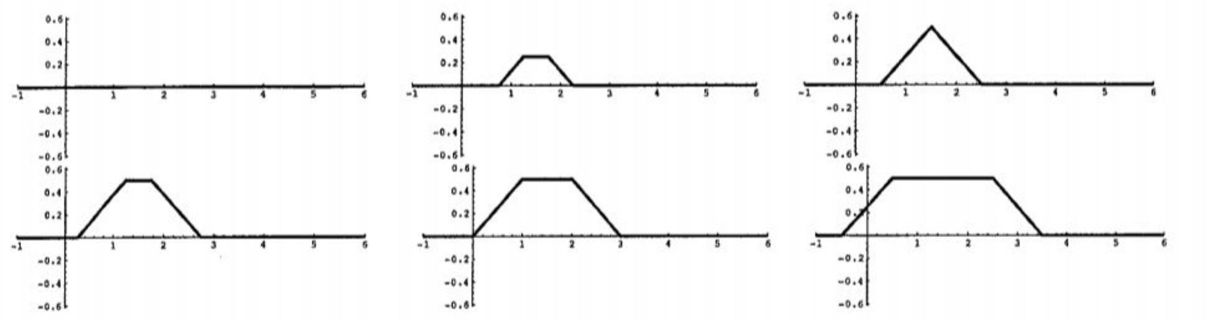
2.4.3

(a) The solution initially forms a trapezoidal displacement, with linearly growing height and sides of slope $\pm .5$ expanding in both directions from 1 and 2 at unit speed. At time $t = .5$, the height reaches .5, and it momentarily forms a triangle. After this the diagonal sides propagate to the right and to the left with unit speed, as the .5 displacement between then grows in extent.

$$\text{For } t < \tfrac{1}{2}\text{, we have } u(t,x) = \begin{cases} \frac{1}{2}(x-1+t), & 1-t < x < 1+t, \\ t & 1+t < x < 2-t, \\ \frac{1}{2}(2+t-x), & 2-t < x < 2+t, \\ 0, & \text{otherwise}, \end{cases}$$

$$\text{For } t \geq \tfrac{1}{2}\text{, we have } u(t,x) = \begin{cases} \frac{1}{2}(x-1+t), & 1-t < x < 2-t, \\ \frac{1}{2} & 2-t < x < 1+t, \\ \frac{1}{2}(2+t-x), & 1+t < x < 2+t, \\ 0, & \text{otherwise}, \end{cases}$$

(b) Plotted at times $t = 0, ..25.5, .75, 1., 1.5$:



2.4.4b, d

$$(b)\ \frac{1}{2}\int_{x-t}^{x+t} 2\cos(2z)\,dz = \frac{\sin 2(x+t) - \sin 2(x-t)}{2}; \quad \star\ (d)\ \frac{(x+t)^2 + (x-t)^2}{2}.$$

2.4.10

Solution: The solution to the initial value problem is

$$u(t,x) = \begin{cases} \dfrac{\omega \sin 2t - 2\sin \omega t}{2(\omega^2-4)} \cos x, & \omega \neq \pm 2, \\ \frac{1}{8}(\sin 2t - 2t\cos 2t)\cos x, & \omega = \pm 2. \end{cases}$$

Thus,

$$g(t) = u(t,0) = \begin{cases} \dfrac{\omega \sin 2t - 2\sin \omega t}{2(\omega^2-4)}, & \omega \neq \pm 2, \\ \frac{1}{8}(\sin 2t - 2t\cos 2t), & \omega = \pm 2, \end{cases}$$

is periodic when $\omega \neq \pm 2$ is a rational number, quasi-periodic when ω is irrational, and non-periodic and resonant when $\omega = \pm 2$.

2.4.11

Solution: (*a*) $u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t)$; (*b*) True;
(*c*) $u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4} - \frac{1}{4}\cos 2t$.
(*d*) The solution remains bounded and periodic, and hence is not resonant.
(*e*) Now the solution is $u(t,x) = \frac{1}{4}\sin(x-2t) + \frac{3}{4}\sin(x+2t) + \frac{1}{4}t - \frac{1}{4}\sin 2t$. In this case, the solution is no longer periodic or bounded, and hence a form of resonance is exhibited.

2.4.13

First of all, the decay assumption implies that $E(t) < \infty$ for all t. To show $E(t)$ is constant, we prove that its derivative is 0. Using the smoothness of the solution to justify bringing the derivative under the integral sign, we compute

$$\frac{dE}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty}\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2u_x^2\right)dx = \int_{-\infty}^{\infty}\left(u_t u_{tt} + c^2 u_x u_{xt}\right)dx$$
$$= c^2\int_0^{\ell}\left(u_t u_{xx} + u_x u_{xt}\right)dx = c^2\int_{-\infty}^{\infty}\frac{d}{dx}\left(u_t u_x\right)dx = 0,$$

since $u_t, u_x \to 0$ as $x \to \infty$. *Q.E.D.*

2.4.15

(*a*) As in Exercise 2.4.13, we compute

$$\frac{dE}{dt} = \int_{-\infty}^{\infty}\left(u_t u_{tt} + c^2 u_x u_{xt}\right)dx = \int_{-\infty}^{\infty}\left[\,c^2(u_t u_{xx} + u_x u_{xt}) - a\,u_t^2\,\right]dx$$
$$= c^2\int_{-\infty}^{\infty}\frac{d}{dx}\left(u_t u_x\right)dx - a\int_{-\infty}^{\infty}u_t^2\,dx = -\,\beta\int_{-\infty}^{\infty}u_t^2\,dx \le 0,$$

since $a > 0$. Thus, $E(t)$ is a nonincreasing function of t.

(*b*) First, let $u(t,x)$ be the solution to the initial-boundary value problem with zero initial conditions, and hence zero initial energy: $E(0) = 0$. Since $0 \le E(t) \le E(0)$ is decreasing, and nonnegative, we conclude that $E(t) \equiv 0$. But since the energy integrand is nonnegative, this can only happen if $u_t = u_x = 0$ for all (t,x), and hence $u(t,x)$ must be a constant function. Moreover, its initial value is $u(0,x) = 0$, and hence $u(t,x) \equiv 0$. With this in hand, in order to prove uniqueness, suppose $u_1(t,x)$ and $u_2(t,x)$ are two solutions to the initial-boundary value problem. Then, by linearity, their difference $u(t,x) = u_1(t,x) - u_2(t,x)$ solves the homogeneous initial-boundary value problem analyzed in part (a), and so must be identically zero: $u(t,x) \equiv 0$. This implies $u_1(t,x) = u_2(t,x)$ for all (t,x), and hence there is at most one solution.