# Homework 3 Solutions

## 3.1.2

(a) $\exp(-n^2 t)\sin n x$ for $n = 1, 2, \ldots$.

(b) $\exp\left[-\left(n+\frac{1}{2}\right)^2 t\right]\sin\left(n+\frac{1}{2}\right)x$ for $n = 0, 1, 2, \ldots$.

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## 3.1.5

5. (a)

| $\lambda$ | Eigenfunctions | Eigensolutions |
|---|---|---|
| $\lambda = -\omega^2 - 1 < -1$ | $\cos\omega x,\ \sin\omega x$ | $e^{-(\omega^2+1)t}\cos\omega x,\ e^{-(\omega^2+1)t}\sin\omega x$ |
| $\lambda = -1$ | $1,\ x$ | $e^{-t},\ e^{-t}x$ |
| $\lambda = \omega^2 - 1 > -1$ | $e^{\omega x},\ e^{-\omega x}$ | $e^{(\omega^2-1)t+\omega x},\ e^{(\omega^2-1)t-\omega x}$ |

(b) $e^{-t},\ e^{-(n^2+1)t}\cos n x,\ e^{-(n^2+1)t}\sin n x,$ for $n = 1, 2, 3, \ldots$.

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## 3.2.1d

★ (d) $\dfrac{\pi^2}{3} + 4\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{\cos kx}{k^2}$

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## 3.2.2d

(d) $\dfrac{2}{\pi}\displaystyle\sum_{j=0}^{\infty}\frac{(-1)^j\sin(2j+1)x}{(2j+1)^2}$

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## 3.2.3

**Solution:** $\sin^2 x \sim \frac{1}{2} - \frac{1}{2}\cos 2x$ and $\cos^2 x \sim \frac{1}{2} + \frac{1}{2}\cos 2x$.

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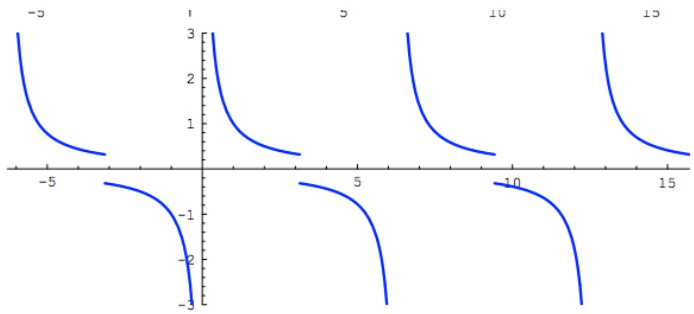
**3.2.6h**

(h) 

not continuous.

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**3.2.9**

(a)

$$\int_a^{a+\ell} f(x)\,dx = \int_0^{\ell} f(x)\,dx - \int_0^{a} f(x)\,dx + \int_{\ell}^{a+\ell} f(x)\,dx. \qquad (*)$$

But, applying the change of variables $y = x - \ell$,

$$\int_{\ell}^{a+\ell} f(x)\,dx = \int_0^{a} f(y+\ell)\,dy = \int_0^{a} f(y)\,dy,$$

which follows from the periodicity of $f$. Thus, the second and third integrals in $(*)$ cancel, which establishes the result. *Q.E.D.*

(b) Using the change of variables $y = x + a$ and part (a),

$$\int_0^{\ell} f(x+a)\,dx = \int_a^{a+\ell} f(y)\,dy = \int_0^{\ell} f(x)\,dx.$$

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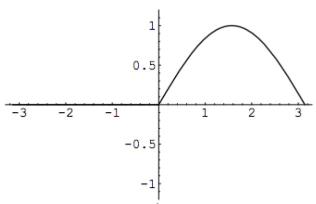
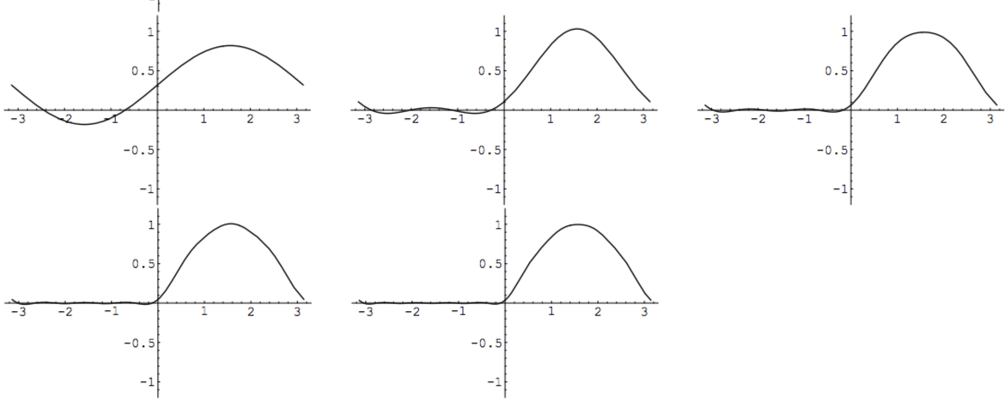
**3.2.25**

(a) 


(b) $f(x) \sim \dfrac{1}{\pi} + \dfrac{1}{2}\sin x - \dfrac{2}{\pi}\displaystyle\sum_{j=1}^{\infty} \dfrac{\cos 2jx}{4j^2-1}$.

(c) 


The maximal errors on $[-\pi,\pi]$ are, respectively .3183, .1061, .06366, .04547, .03537, .02894.

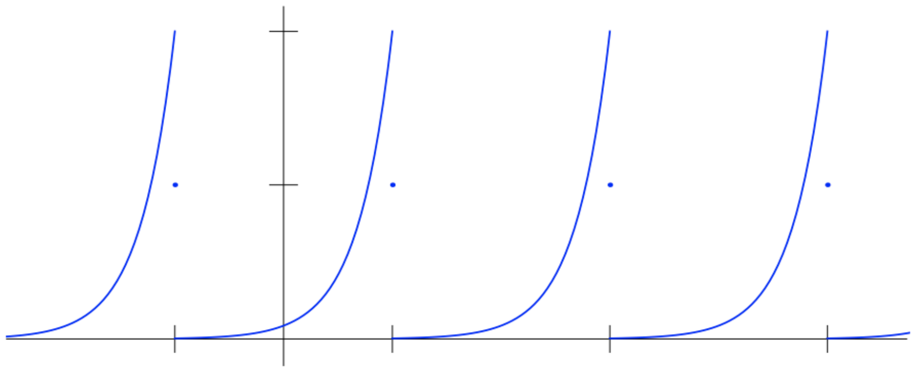
(d) The Fourier series converges (uniformly) to $\sin x$ when $2k\pi \le x \le (2k+1)\pi$ and to 0 when $(2k-1)\pi \le x \le 2k\pi$ for $k = 0, \pm 1, \pm 2, \ldots$.

### 3.2.27

(a) $$e^x \sim \frac{\sinh\pi}{\pi} + \frac{2\sinh\pi}{\pi}\sum_{k=1}^{\infty}(-1)^k\,\frac{\cos k\,x - k\sin k\,x}{1+k^2}.$$

(b) The Fourier series converges for all real $x$ to the $2\pi$–periodic extension of $e^x$, with values $\cosh\pi = \frac{1}{2}(e^{\pi}+e^{-\pi})$ at the discontinuities at $x = \pm\pi, \pm3\pi, \ldots$. The convergence is not uniform because the limiting sum is not continuous.

(c)



### 3.2.30

(a) If $\widetilde{f}(x)$ is the $2\pi$–periodic extension of $f(x)$, then the Fourier series converges to $\widetilde{f}(2x)$, which is the $\pi$–periodic extension of $f(2x)$.

(b) The Fourier series $2\left(\sin 2x - \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x - \cdots\right)$ converges to the $\pi$–periodic extension of the function $\widehat{f}(x) = 2x$ for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, or, equivalently, the $2\pi$–periodic extension of

$$f(x) = \begin{cases} 2(x+\pi), & -\pi < x < -\frac{1}{2}\pi, \\ 0, & x = \pm\frac{1}{2}\pi, \\ 2x, & -\frac{1}{2}\pi < x < \frac{1}{2}\pi, \\ 2(x-\pi), & \frac{1}{2}\pi < x < \pi. \end{cases}$$