Schur and LLT Polynomials from Lattice Models

Sylvester Zhang

Univ. Minnesota

joint work with: M. Curran C. Yost-Wolff & V. Zhang

mentored by B. Brubaker C. Frechette K. Weber

at UMN Combinatorics REU

Outline

- Symmetric Functions, SSYT & Schur polynomials.
- "Ice" and Schur polynomials
- Ribbon Tableaux and LLT polynomials
- Lattice Model for LLT polynomials.

  ☆ The same (bijectively) model is given independently in arxiv 2012.02376 (Corteel - Gitlin - Keating - Meza)

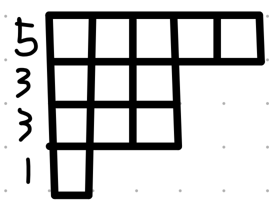
  ☆ The Colored fermionic Vertex model of Aggarwal - Borodin - Wheeler (arxiv 2101.01605) specializes to Macdonald, Non symmetric Macdonald, LLT, factorial LLT ......
- Yang-Baxter Equation

Semi Standard Young Tableaux

- We say $\lambda = (\lambda_1\, \lambda_2 \cdots \lambda_k)$ is a partition of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$

- They can be represented by Young diagrams :

$$\lambda = (5,3,3,1)$$



- Young Tableaux are filling of a Young diagram with integers.
A tableau is called semi-standard if row entries are weakly increasing and column entries are strickly increasing.
Denote $SSYT_\lambda^n$ the set of all semi-standard Young Tableaux whose shape is λ.

SSYT and Schur Polynomials

- Examples of SSYT

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2}\end{array} \in SSYT^{3}_{(4,3,2)} \qquad \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 \\ \hline 3 & 4 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \in SSYT^{4}_{(4,4,3,1)}$$

- Define a weight on SSYTs : $wt : SSYT^{k}_{\lambda} \to \mathbb{Z}[x_1, \cdots, x_k]$

$$wt(T) = \prod_{i=1}^{k} x_i^{\#\text{ of } i \text{ in } T}$$

e.g. $$wt\left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 \\ \hline 3 & 4 & 4 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \right) = x_1 \, x_2^{3} \, x_3^{4} \, x_4^{4}$$

SSYT and Schur Polynomials

- The **Schur polynomial** of shape λ is defined to be

$$s_\lambda(x_1 \cdots x_k) = \sum_{T \in SSYT_\lambda^k} wt(T)$$

E.g. $\lambda = (3,1)$

$$s_\lambda(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Theorem Schur polynomials are symmetric, i.e.

$$s_\lambda(x_1 \cdots x_k) = s_\lambda(x_{\pi(1)} \cdots x_{\pi(k)}) \text{ for any } \pi \in S_k$$

"permuting the variable doesn't change the polynomial"

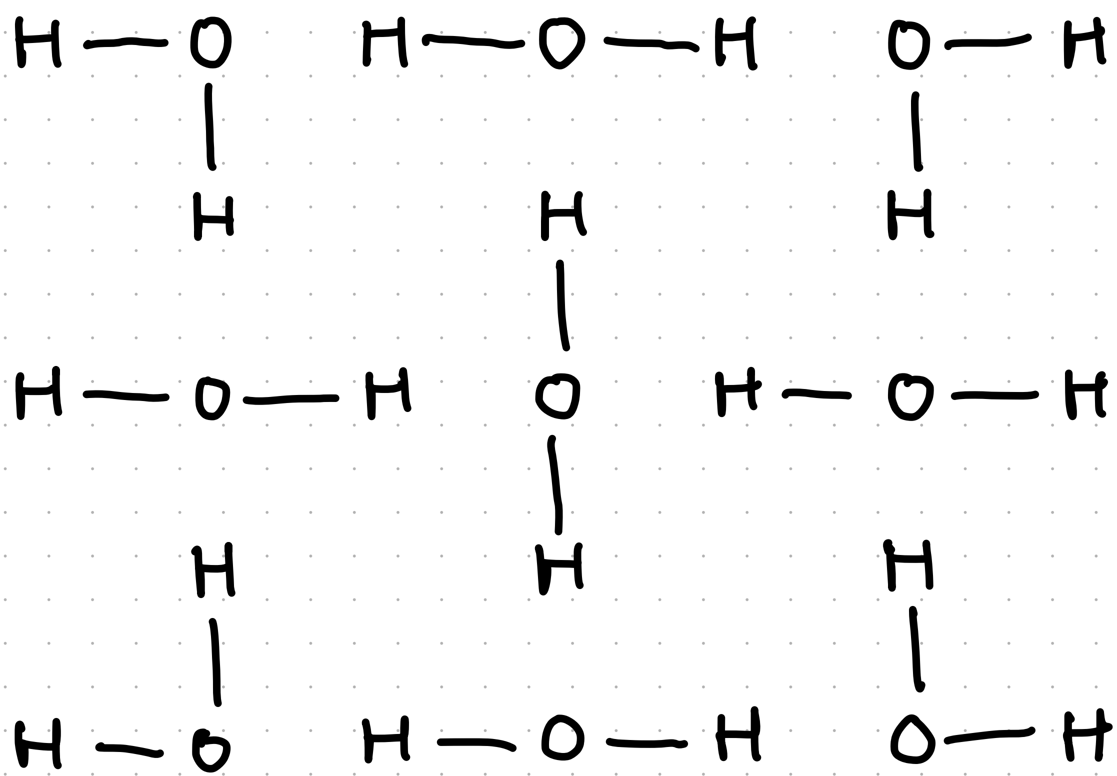
Ice

An **2-dim ice model** is a filling of the following diagram with | and —, so that every Oxygen atom is connected to 2 hydrogen atoms.

H	O	H	O	H	O	H
	H		H		H	
H	O	H	O	H	O	H
	H		H		H	
H	O	H	O	H	O	H

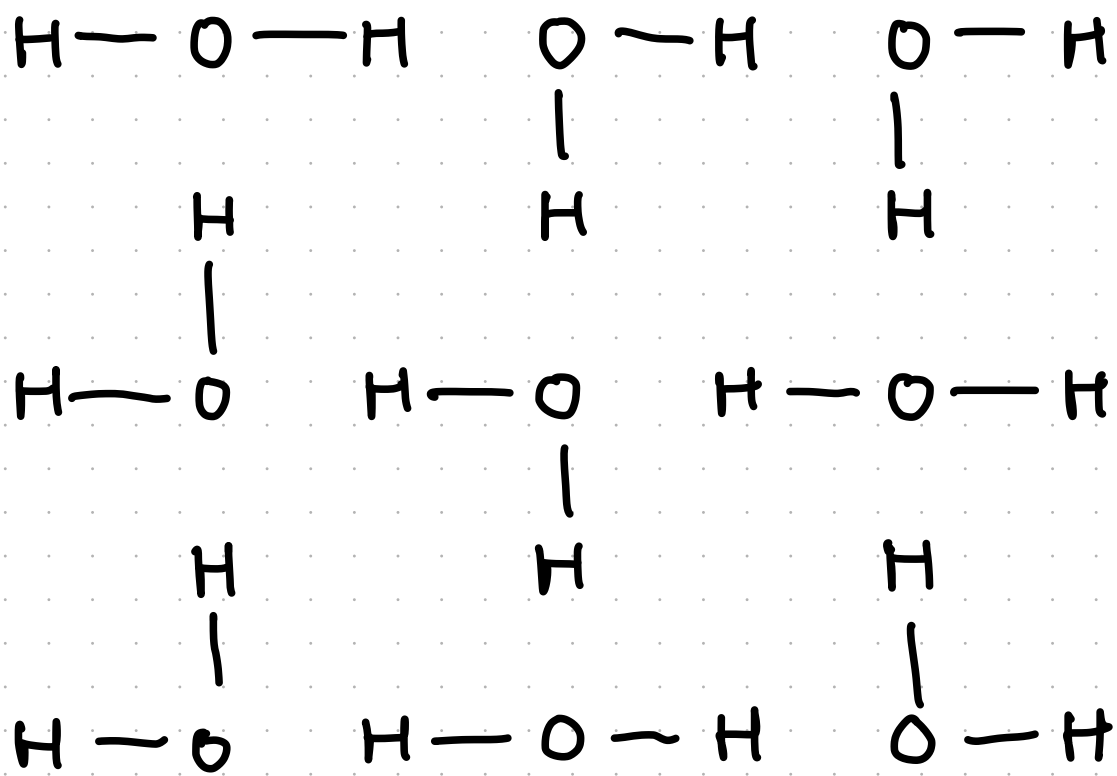
Ice

An **2-dim ice model** is a filling of the following diagram with | and —, so that every Oxygen atom is connected to 2 hydrogen atoms.



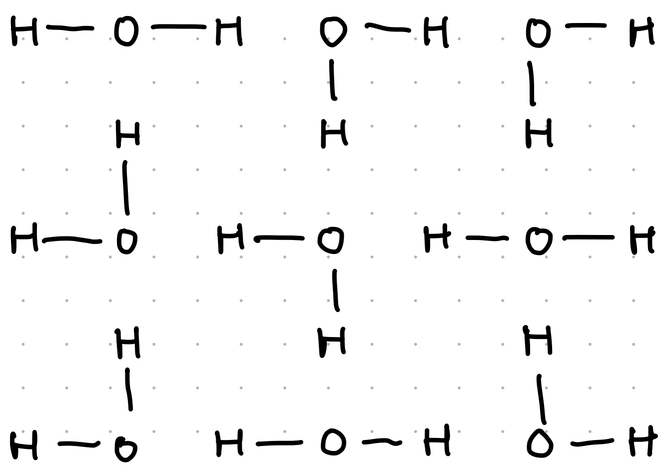
Ice

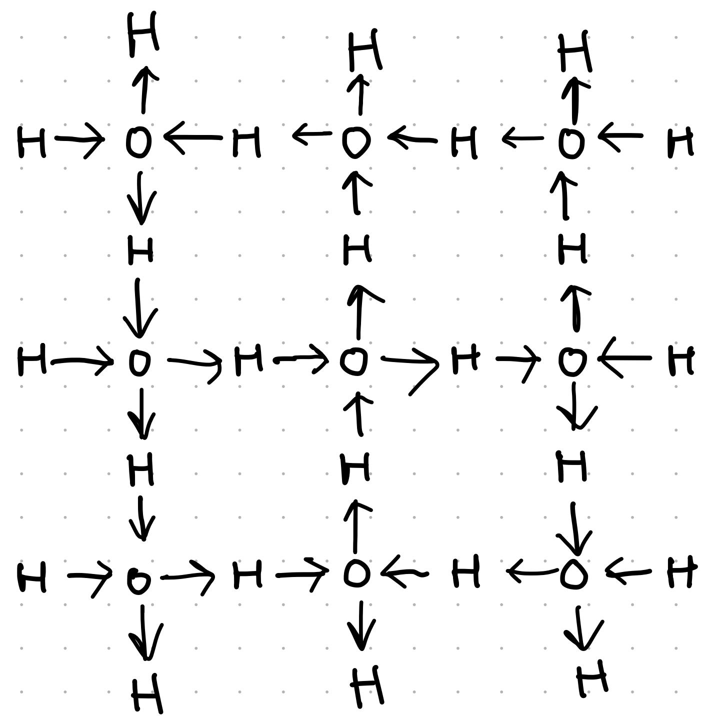
An **2-dim ice model** is a filling of the following diagram with | and —, so that every Oxygen atom is connected to 2 hydrogen atoms.



- Replace every H—O with an arrow H → O

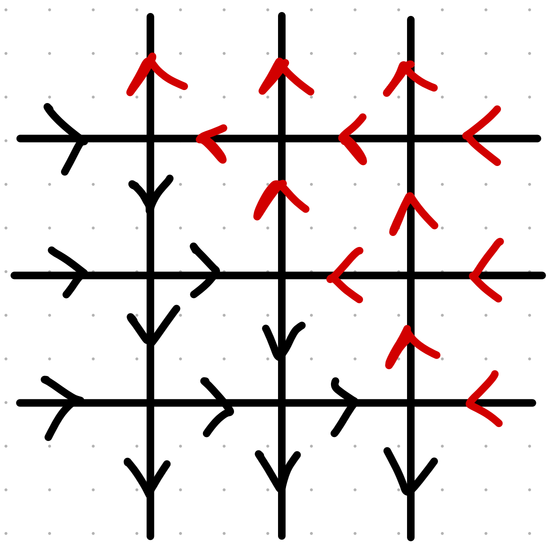
- Fill out the empty spaces with O → H





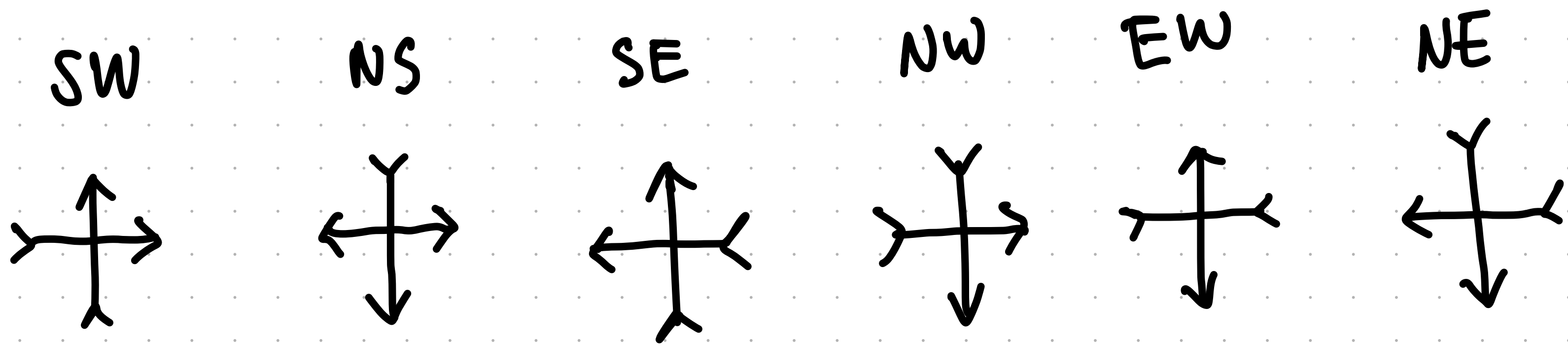
This is a "6-vertex" model.

The 6-vertex model is a configuration of arrows on every edges of a square lattice, so that every vertex has 2 in-arrows and 2 out-arrows



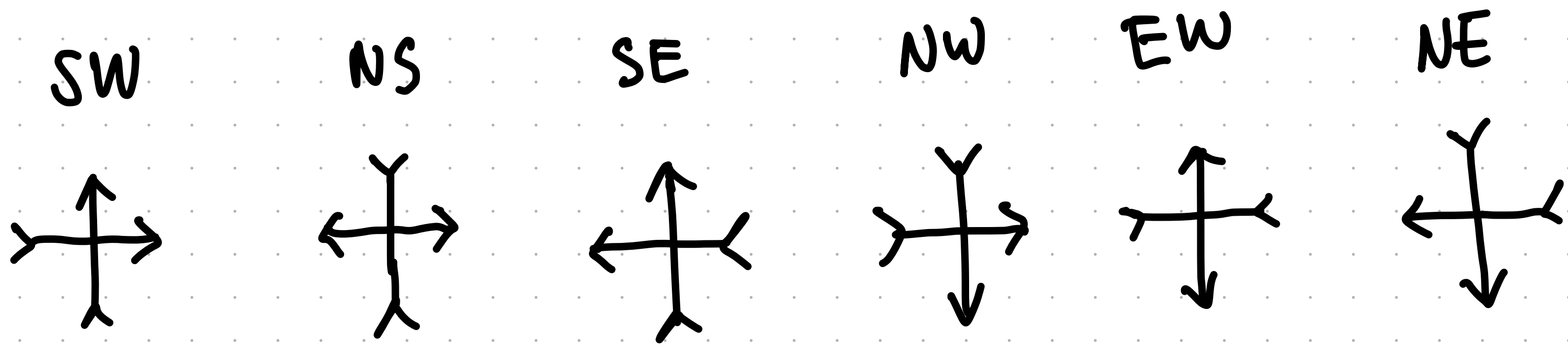
The 6-vertex model is a configuration of arrows on every edges of a square lattice, so that every vertex has 2 in-arrows and 2 out-arrows.

There are 6 possible vertex configuration:

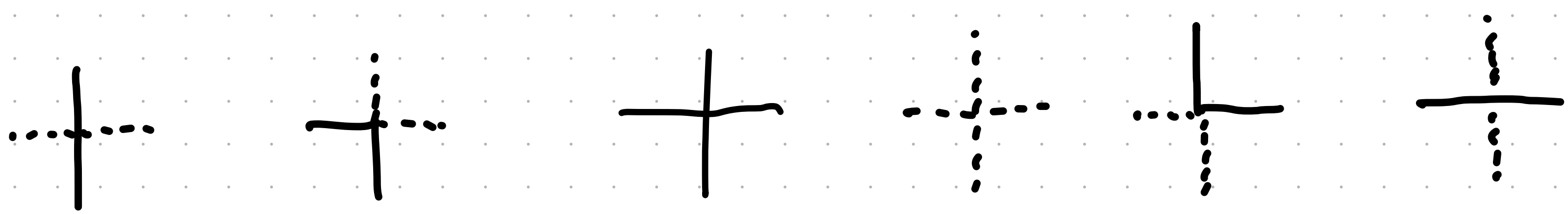


The 6-vertex model is a configuration of arrows on every edges of a square lattice, so that every vertex has 2 in-arrows and 2 out-arrows.

There are 6 possible vertex configuration:

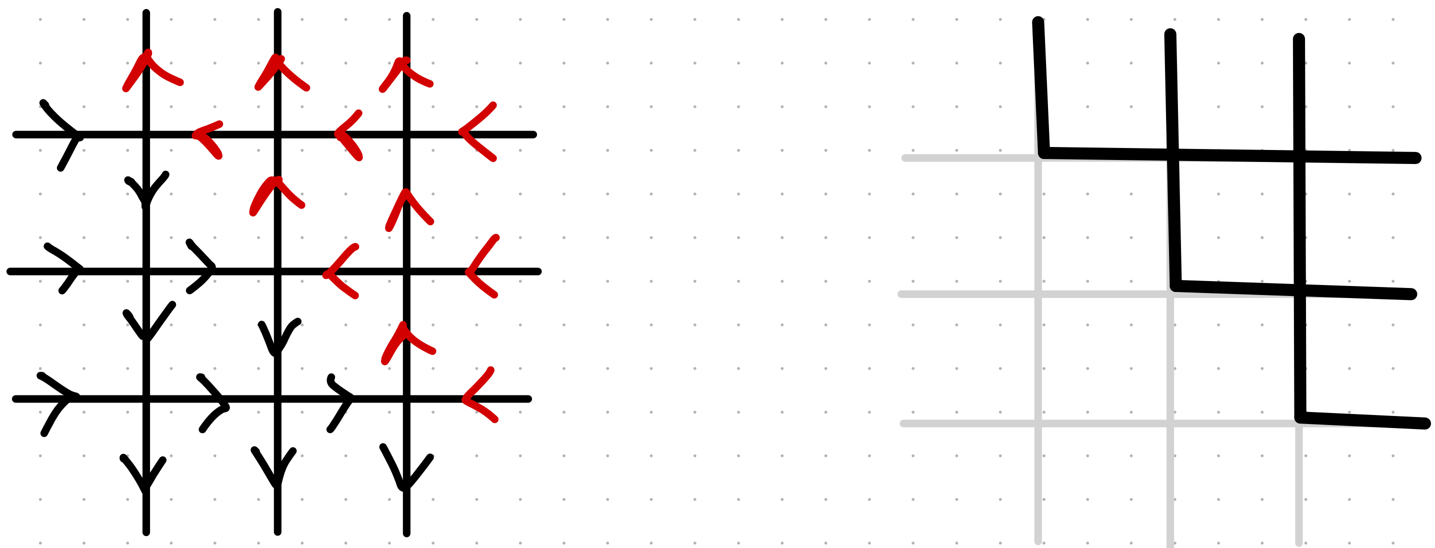


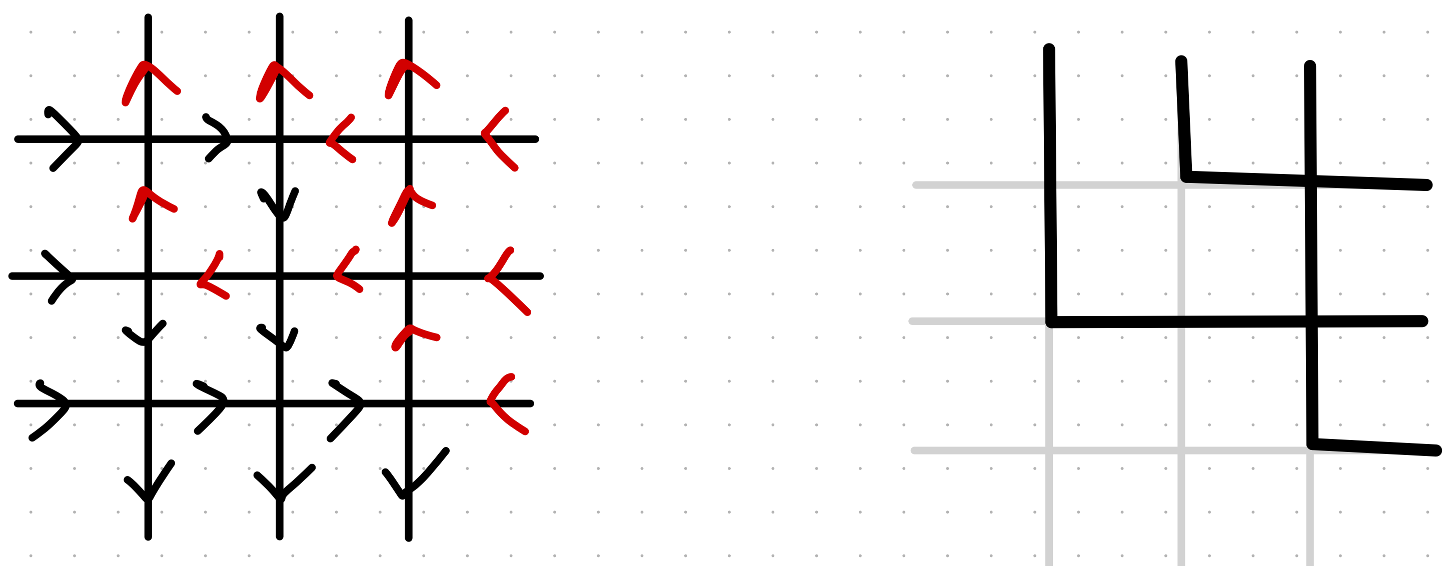
which can be thought of as "lattice paths":



left and up = path ; right and down = no path

The 6 vertex model = Lattice paths

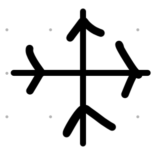
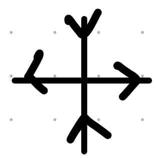
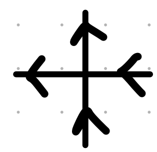
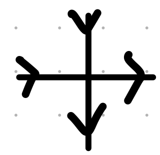
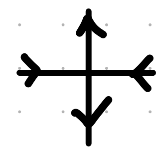
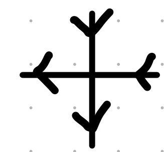




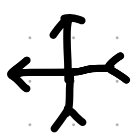
Boltzmann weights and Partition Function

⚠ not the same as integer partition.

For every vertex, define its Boltzman weight as follows

v						
wt(v)	1	1	0	1	x_1	x_i

where i is the row number.

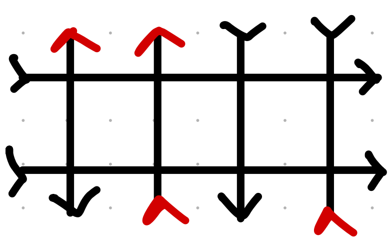
Note that this is actually a 5-vertex model.
because  is unweighted.

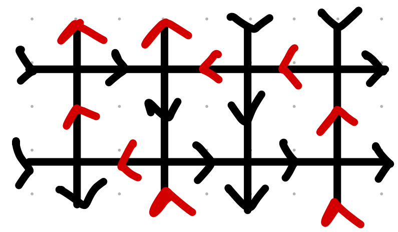
Boltzmann weights and Partition Function

For a given "boundary condition", define the partition function to be

$$P(x_1 \ldots x_k) = \sum_{\substack{\text{admissible} \\ \text{states } T}} \prod_{\substack{v \text{ is a} \\ \text{vertex of } T}} wt(v)$$

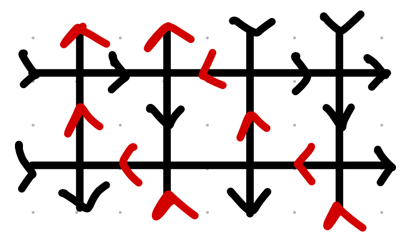
For the following boundary, the partition function is:





$1 \quad x_1 \quad x_1 \quad 1$

$x_2 \quad 1 \quad 1 \quad 1$



$1 \quad x_1 \quad 1 \quad 1$

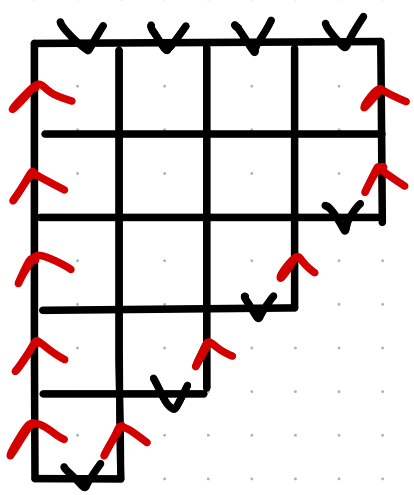
$x_2 \quad 1 \quad x_2 \quad 1$

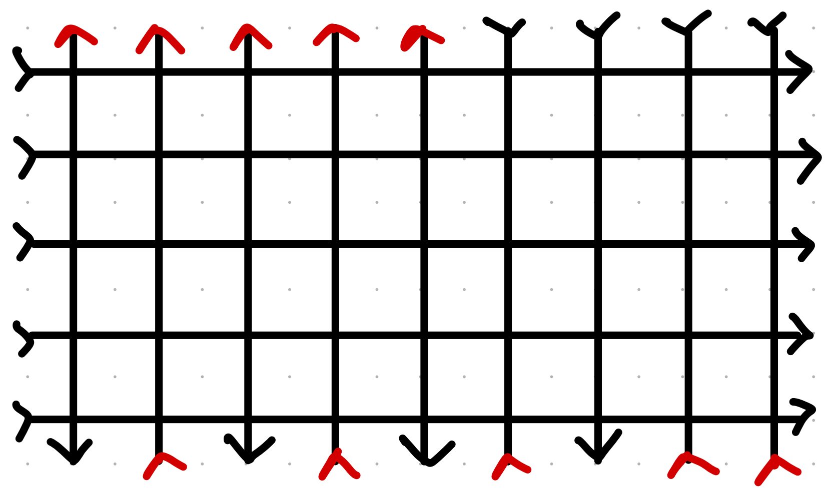
$$P(x_1\, x_2) = x_1^2 x_2 + x_1 x_2^2$$

Boltzmann weights and Partition Function

Boundary Condition:

For a integer partition λ, define a boundary condition:





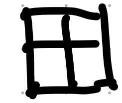
Theorem 1 The partition function under this boundary condition equals the Schur polynomial:

$$P_\lambda(x_1 \cdots x_k) = S_\lambda(x_1 \cdots x_k)$$

Ribbon Tableaux & LLT polynomials

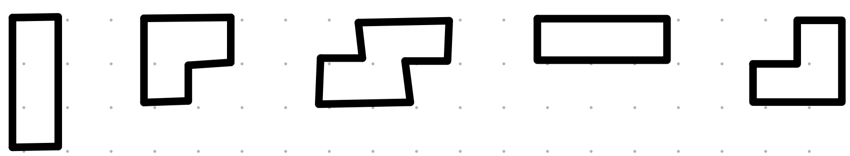
Leclerc, Lascoux, Thibon

"Ribbon Tableaux, Hall-Littlewood Functions, Quantum Affine Algebras, and Unipotent Varieties"
(arXiv 9512031)

A ribbon is a (skew) Young diagram that doesn't contain 

The spin of a ribbon is height − 1.

E.g.

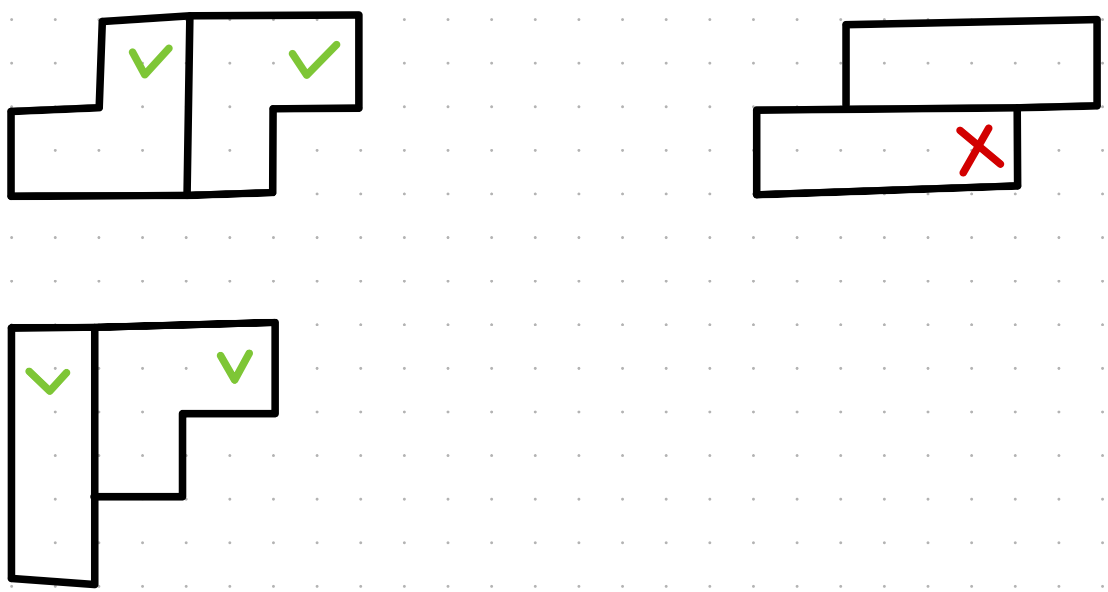


Spin: 2 1 1 0 1

Ribbon Tableaux & LLT polynomials

A **n-horizontal strip** is a tiling of a skew Young diagram by n-ribbons such that the top-right corner of each ribbon touches the northern boundary.

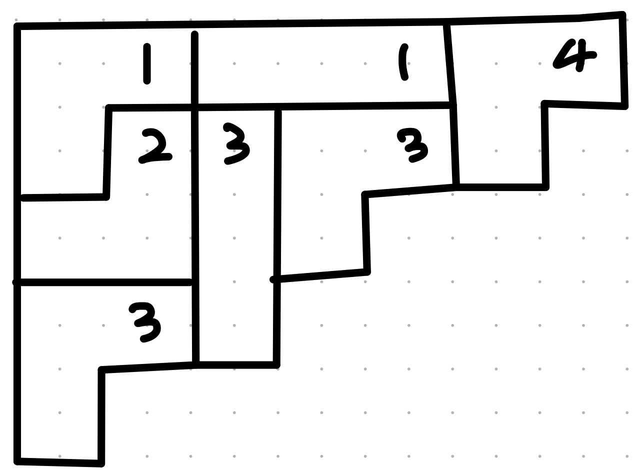
E.g.



Semi Standard Ribbon Tableaux

A SSRT is defined analagously to the SSYTs, with boxes replaced by ribbons, such that the restriction to any number is a horizontal strip.

E.g.



In other words, a SSRT is of shape λ is a sequence of partitions $\phi = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_k = \lambda$ such that $\lambda_1 \setminus \lambda_0$, $\lambda_2 \setminus \lambda_1 \cdots$ are horizontal strips.

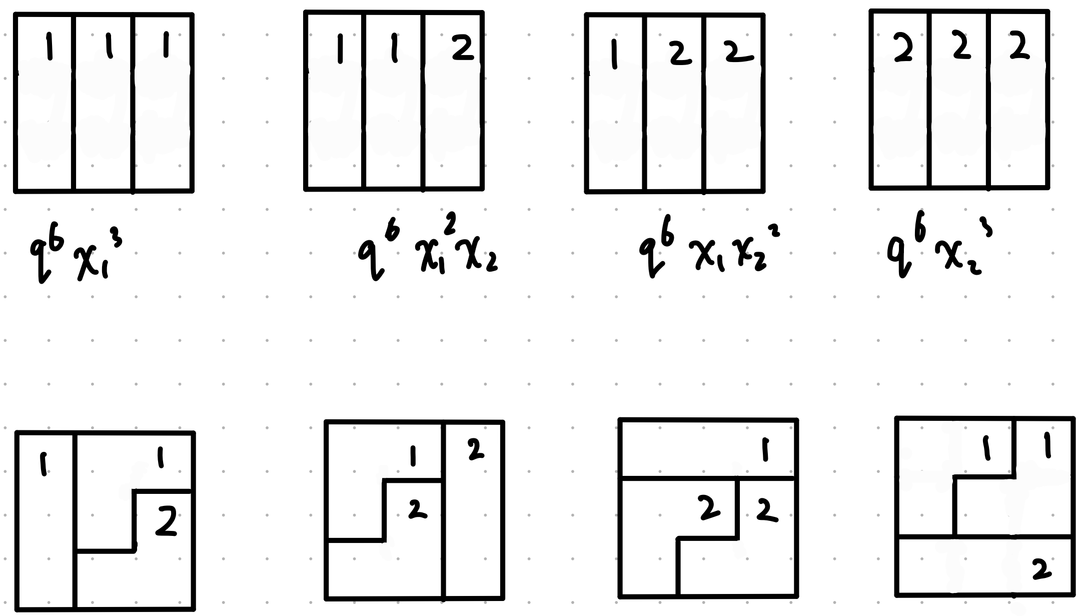
LLT polynomials

Given partition λ (tilable by n-ribbons), define the **n-LLT polynomial** associated to λ to be

$$G_\lambda^{(n)}(x_1 \dots x_k; q) = \sum_{T \in SSRT_\lambda} q^{\mathrm{spin}(T)} \, wt(T)$$

where $\mathrm{spin}(T)$ is the sum of spins of all ribbons in T.

Example of LLT polynomials



$q^6 x_1^3$, $q^6 x_1^2 x_2$, $q^6 x_1 x_2^2$, $q^6 x_2^3$

$q^4 x_1^2 x_2$, $q^4 x_1 x_2^2$, $q^2 x_1 x_2^2$, $q^2 x_1^2 x_2$

$$G^{(3)}_{(3,3,3)}(x_1 x_2 q)$$
$$= q^6(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3) + (q^4 + q^2)(x_1^2 x_2 + x_1 x_2^2)$$

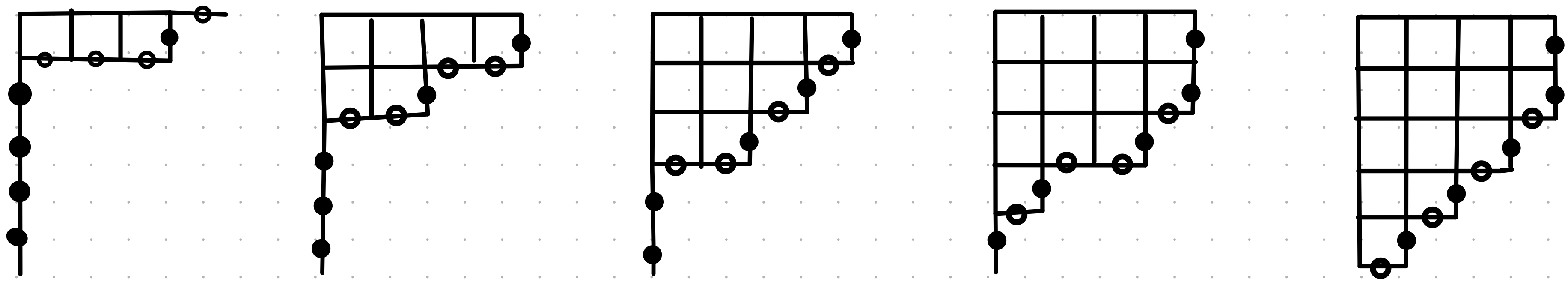
$$G^{(3)}_{(3,3,3)}(x_1 x_2 1)$$
$$= (x_1 + x_2)^3 = S_{\square}(x_1, x_2)^3$$

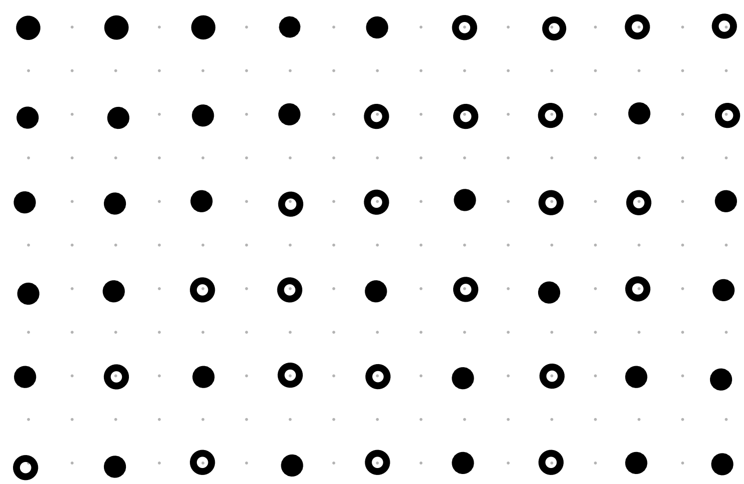
Theorem (LLT)

- LLT polynomials are symmetric.
- when $q=1$, $G^{(n)}_{\lambda}(x_1 \cdots x_k, 1)$ is a product of n Schur polynomials

Young Tableaux and Non intersecting Lattice Paths

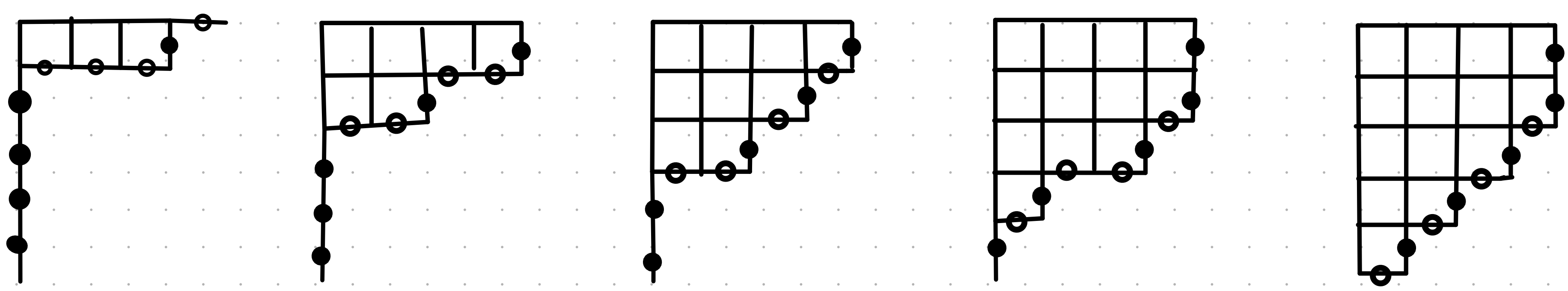
SSYTs are flags of partitions:

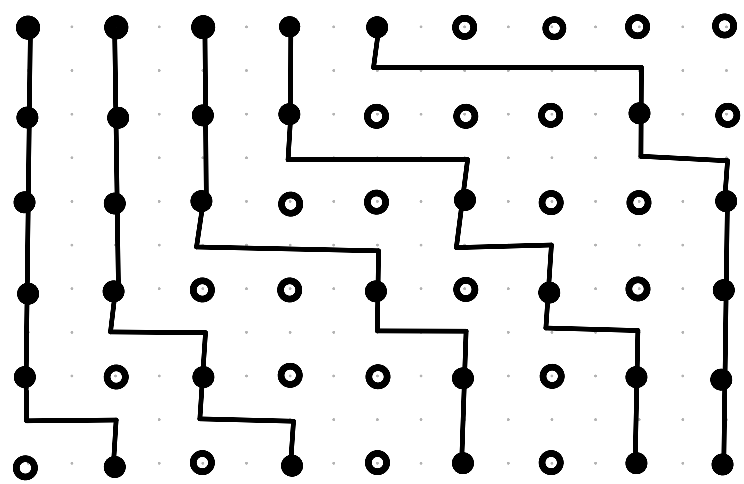




Young Tableaux and Non intersecting Lattice Paths

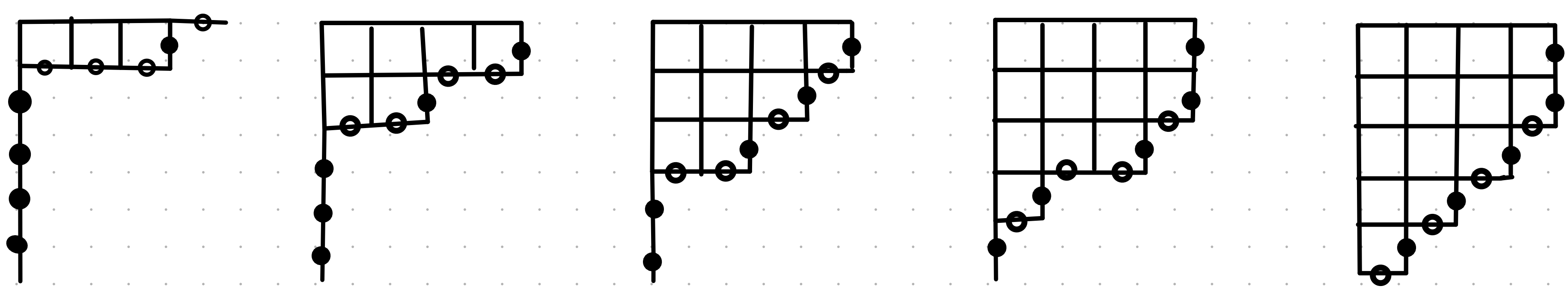
SSYTs are flags of partitions:

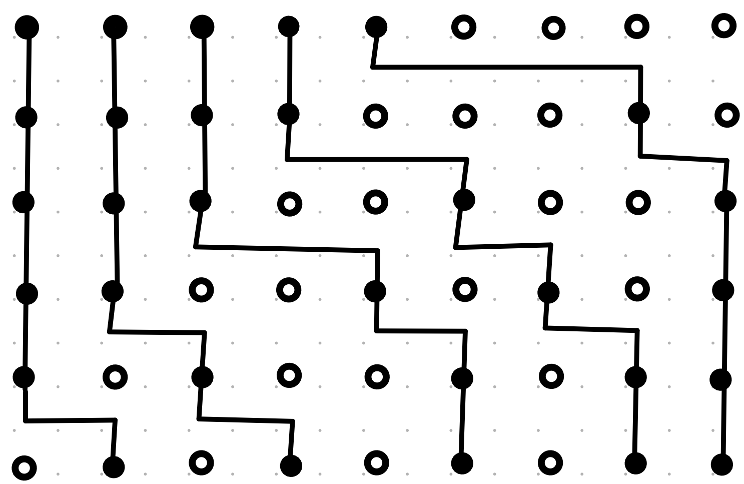


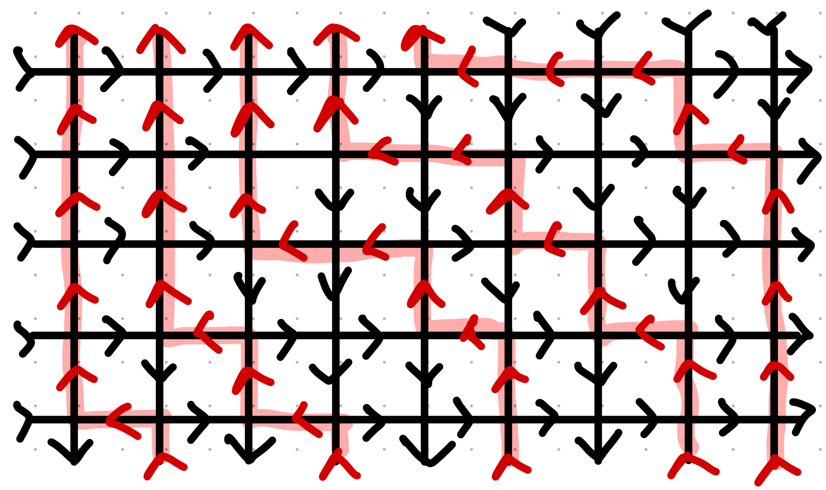


Young Tableaux and Non intersecting Lattice Paths

SSYTs are flags of partitions:

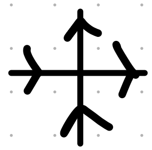
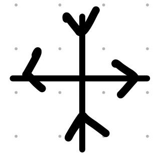
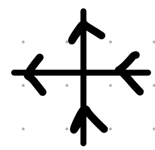
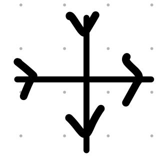
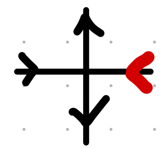
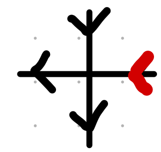






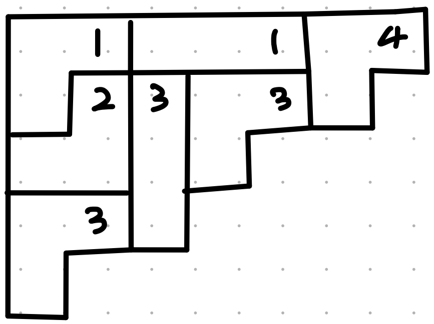
Claim This is a weight preserving bijection.

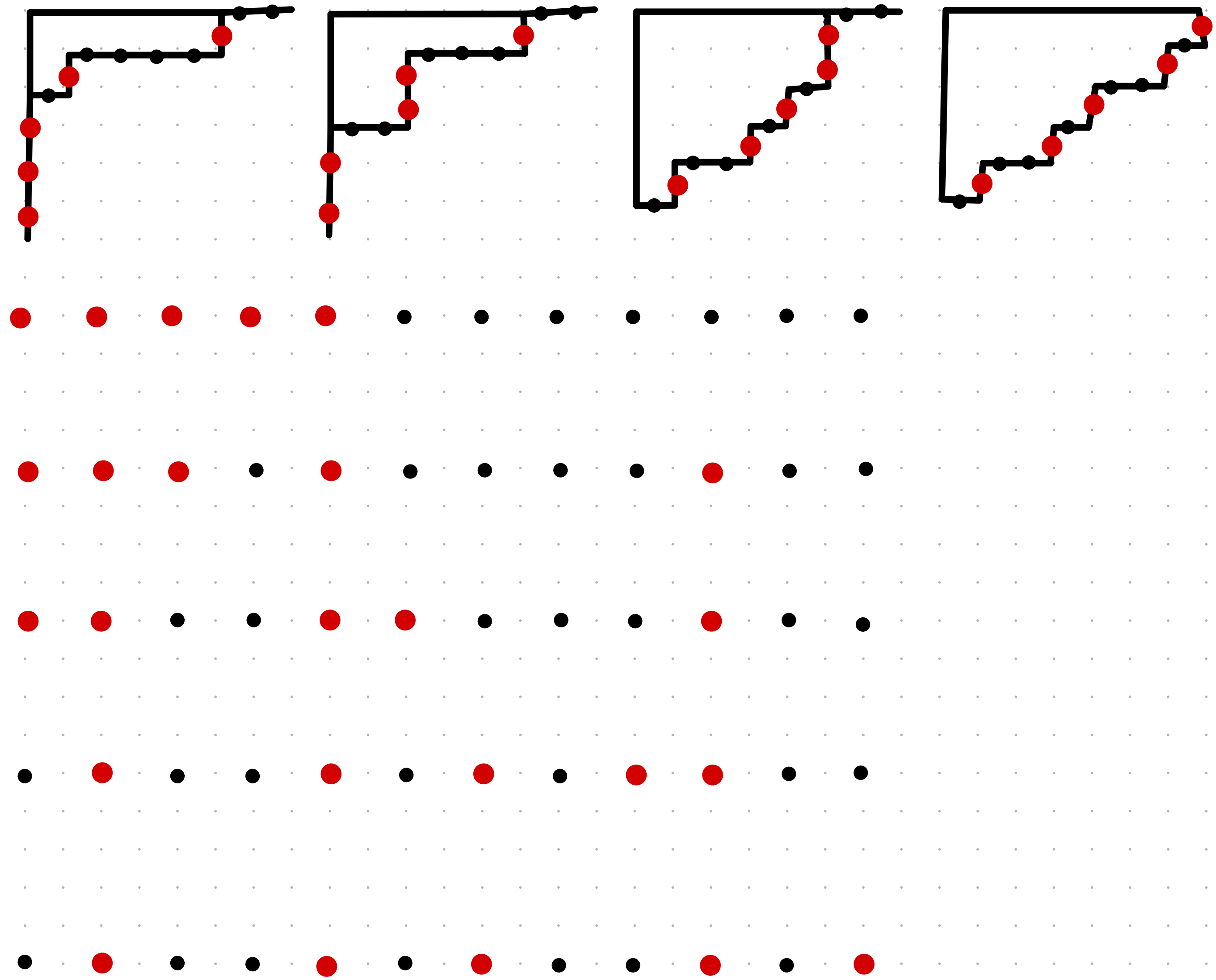
Back to the Bolzmann weights

v						
wt(v)	1	1	0	1	x_1	x_i

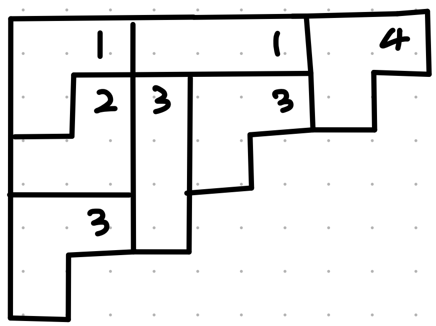
- the 0-weighted vertex is when the paths intersect.
- Every left arrow gets a weight x_i

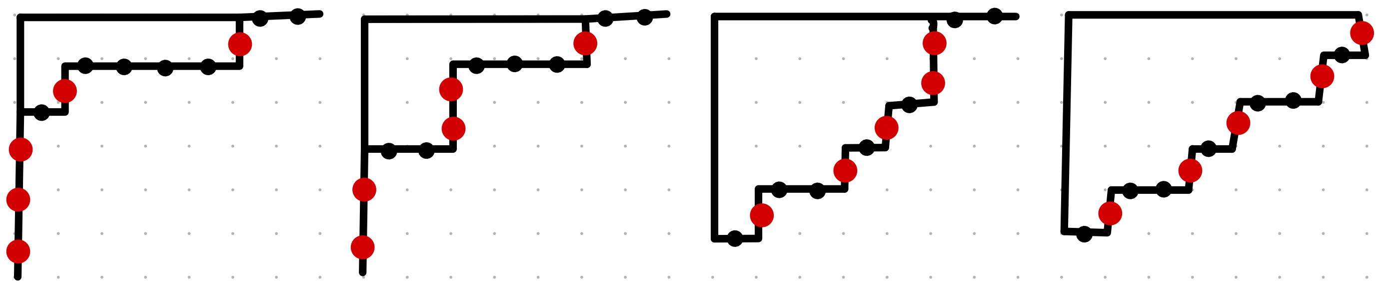
Lattice path for Ribbon Tableaux ??

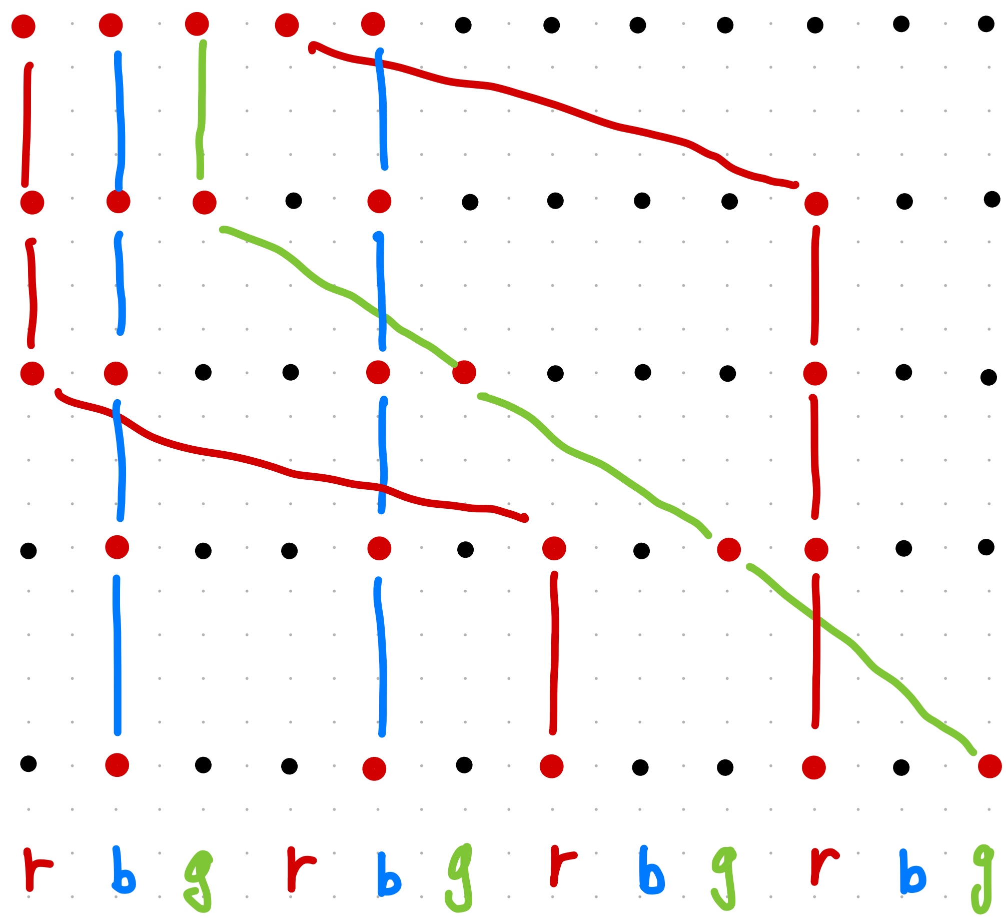




Colored Non-intersecting Lattice Path

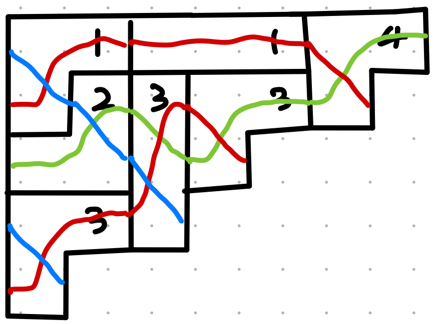


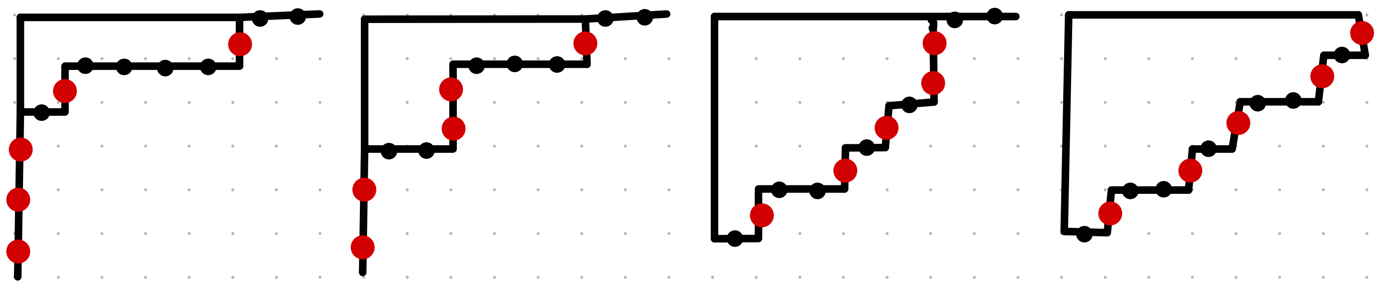


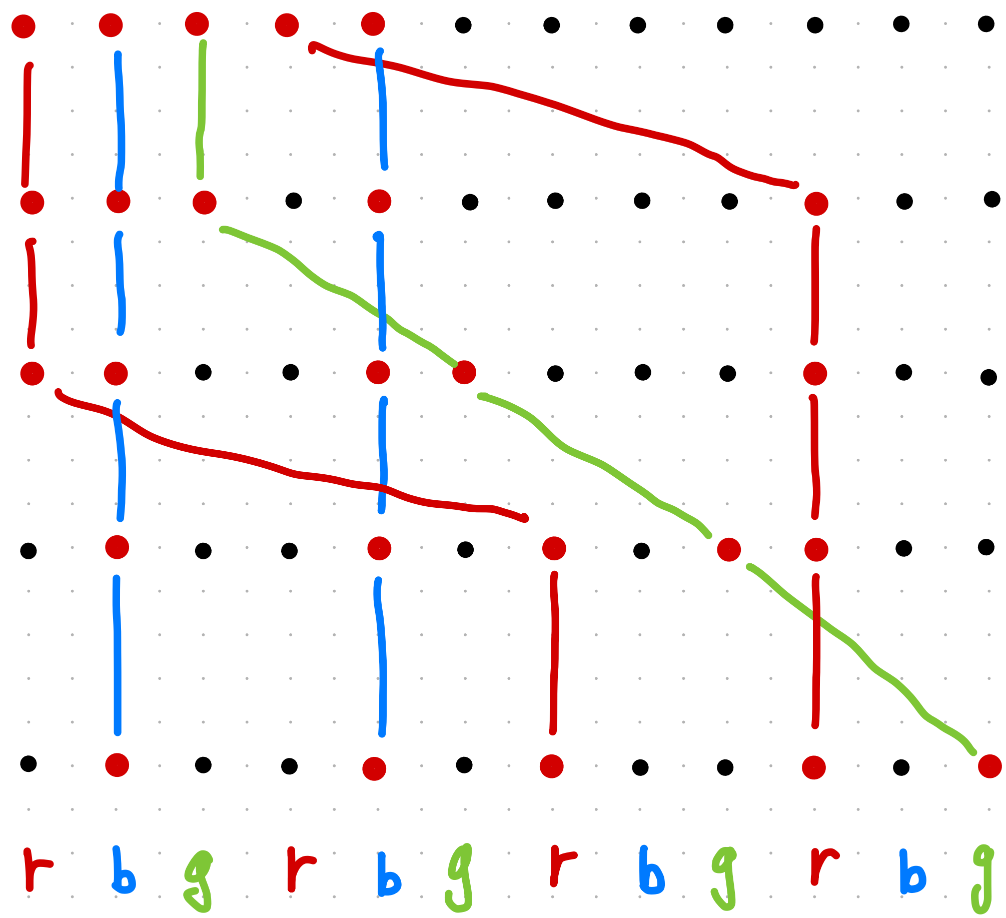


Can't inersect with the same Color.

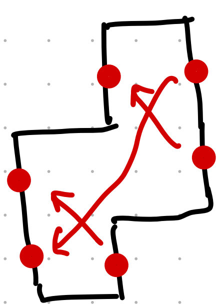
Colored Non-intersecting Lattice Path



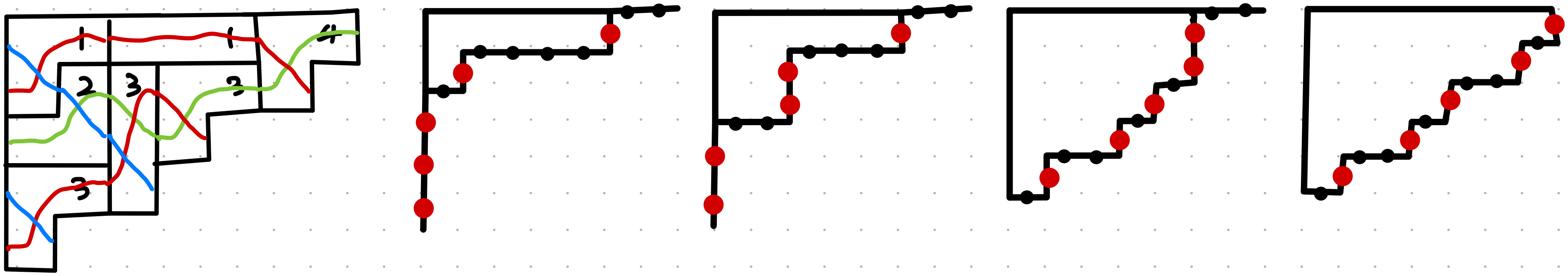


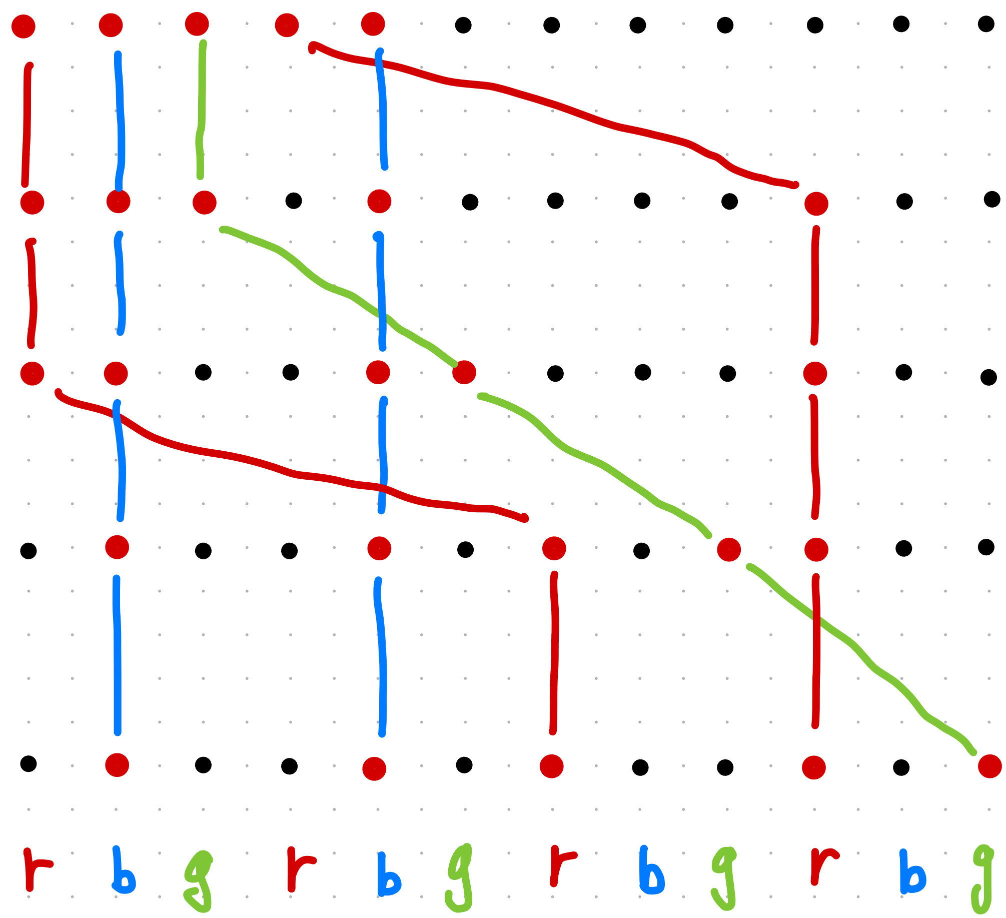


In each ribbon the top-right ● moves to the bottom left, others moves up



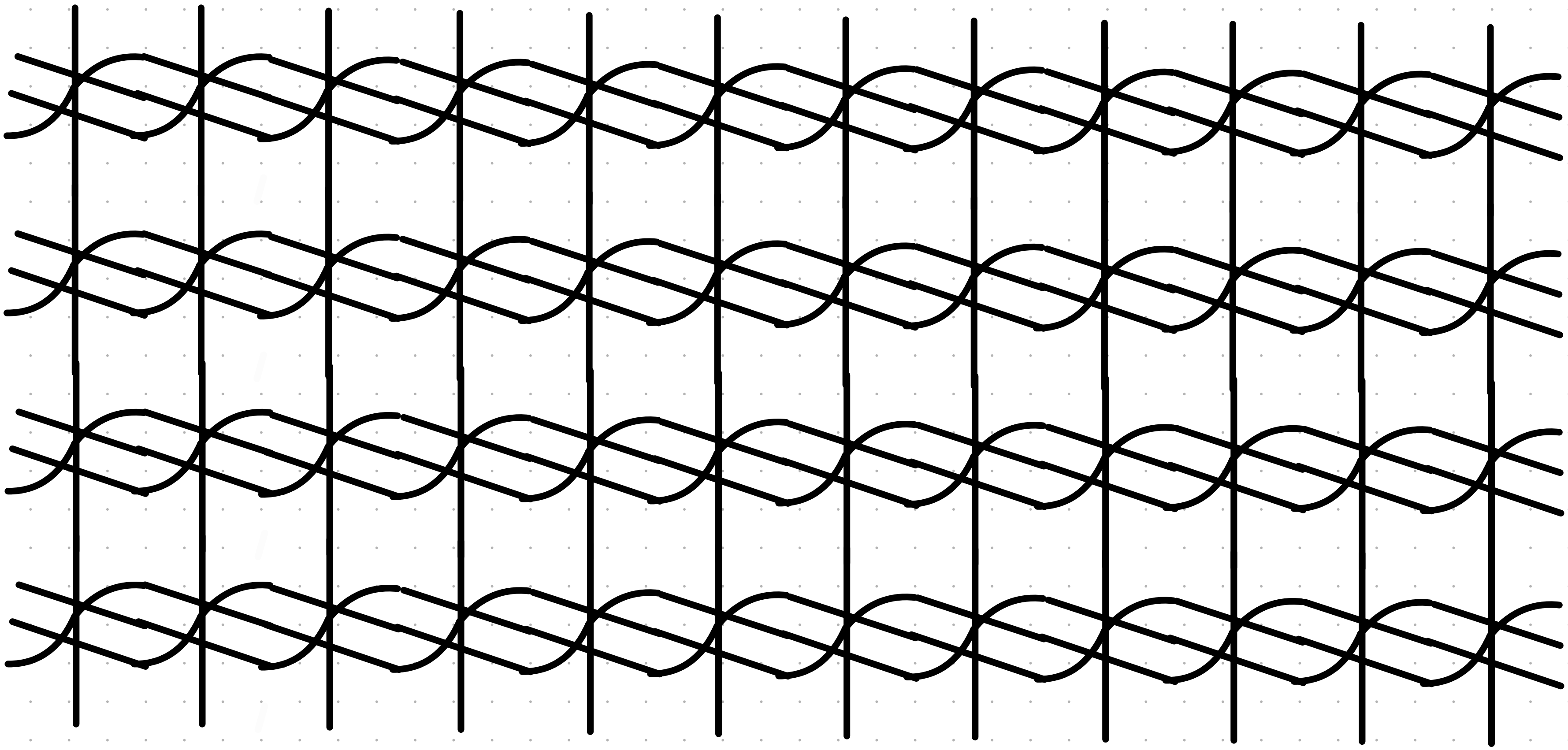
Colored Non-intersecting Lattice Path





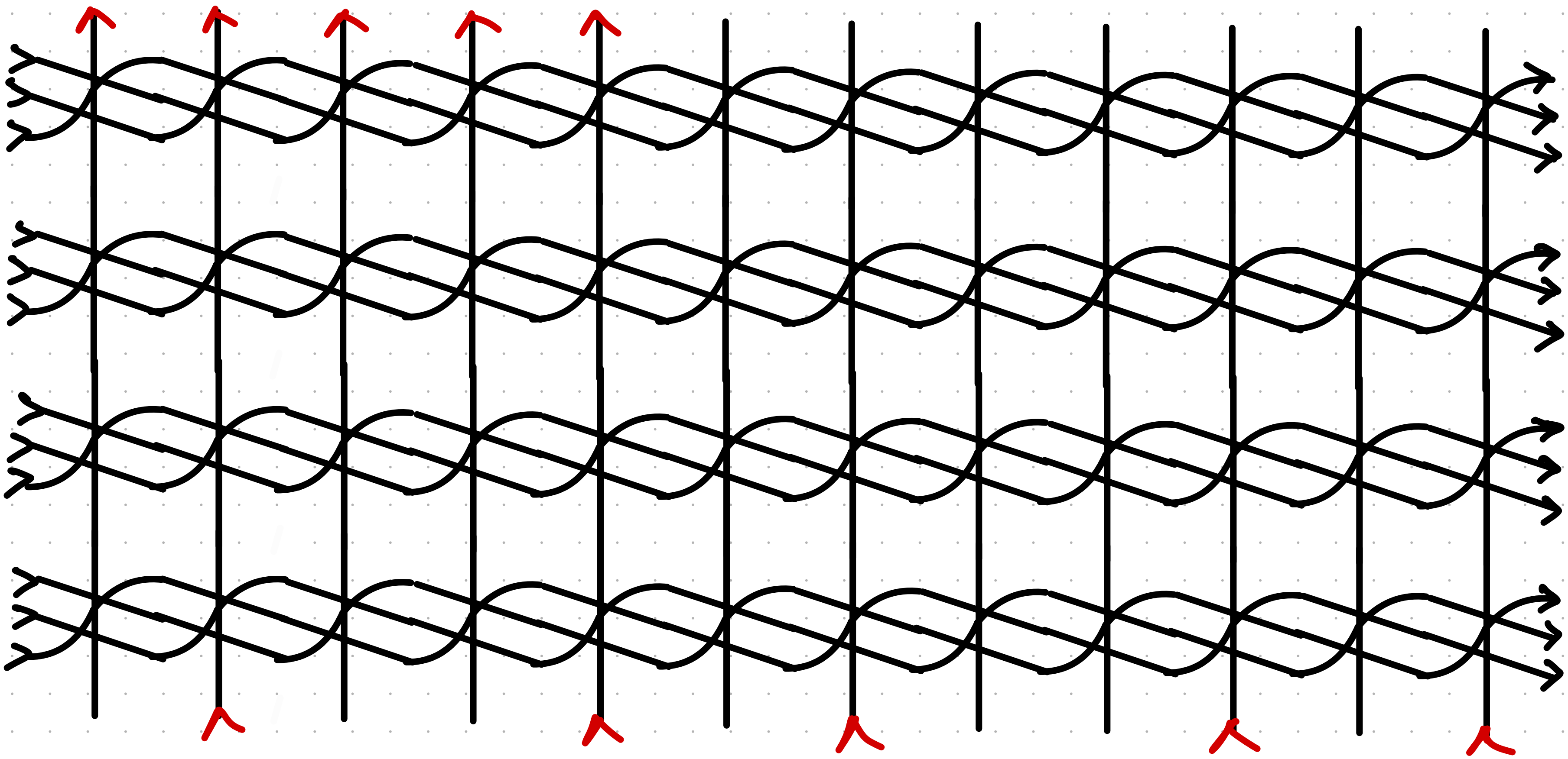
spin = # of intersection

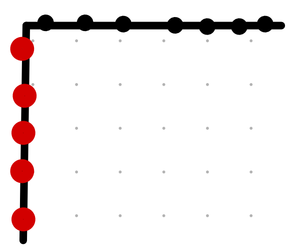
Lattice Model ??

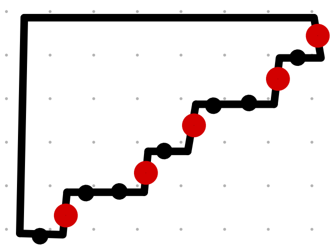


n-ribbon Lattice Model

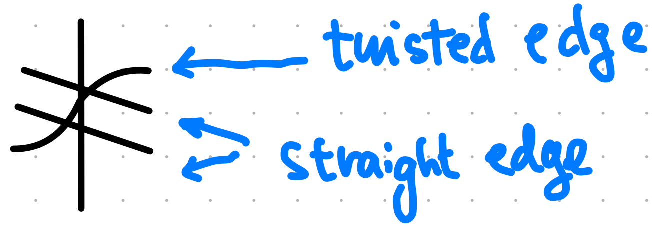
boundary condition





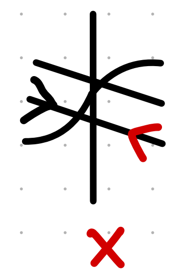


n-ribbon Lattice Model : admissible vertices

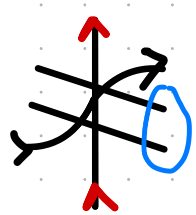
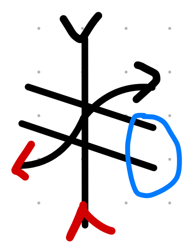
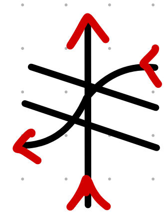
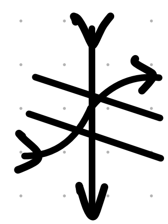
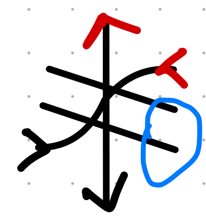
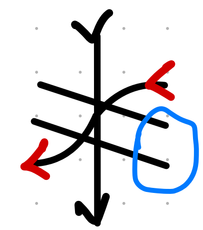


(R1) # of in arrow = # of out arrow

(R2) NO change of arrow on straight edges.

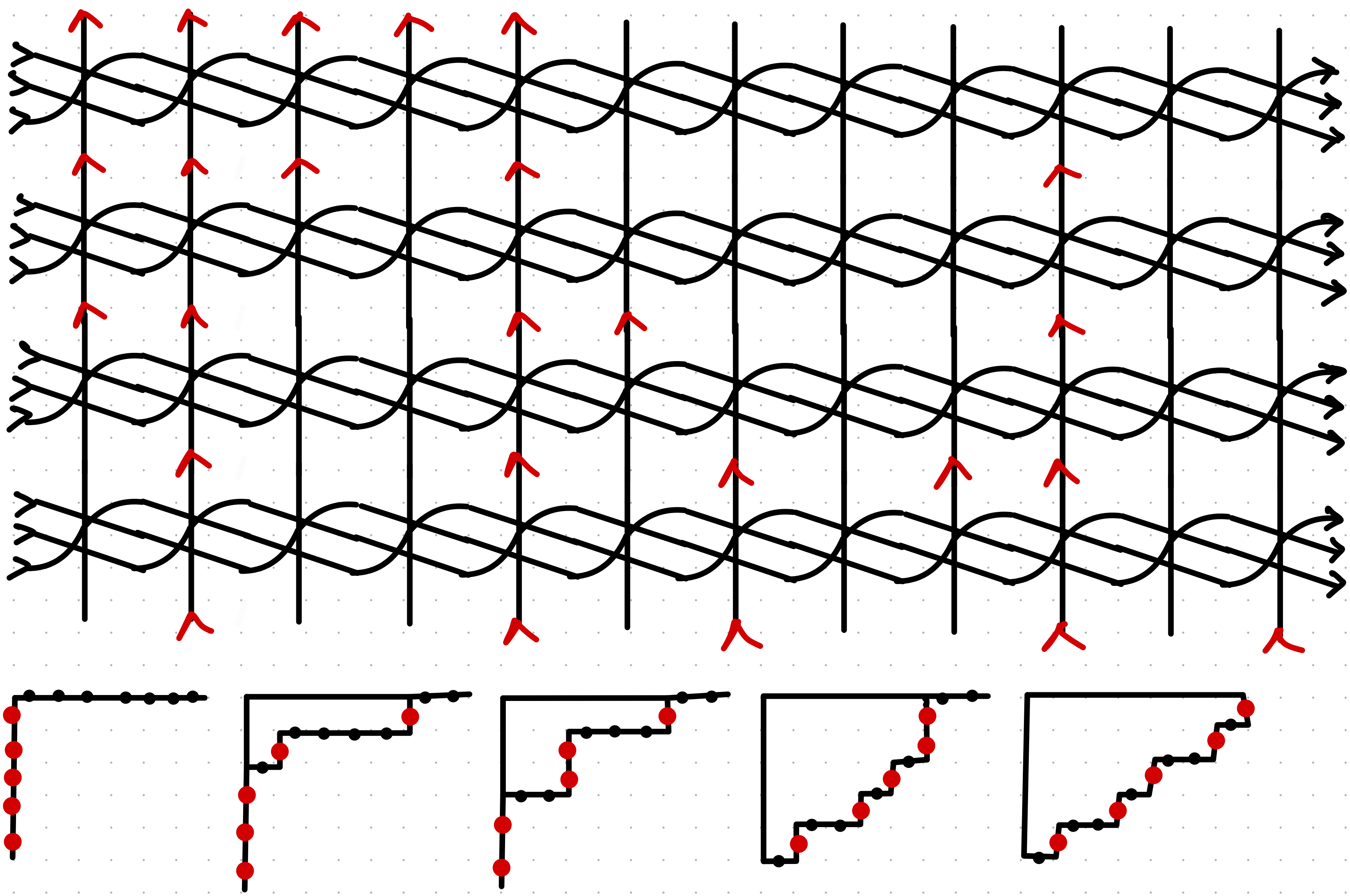


(R3) Bolzmann weights

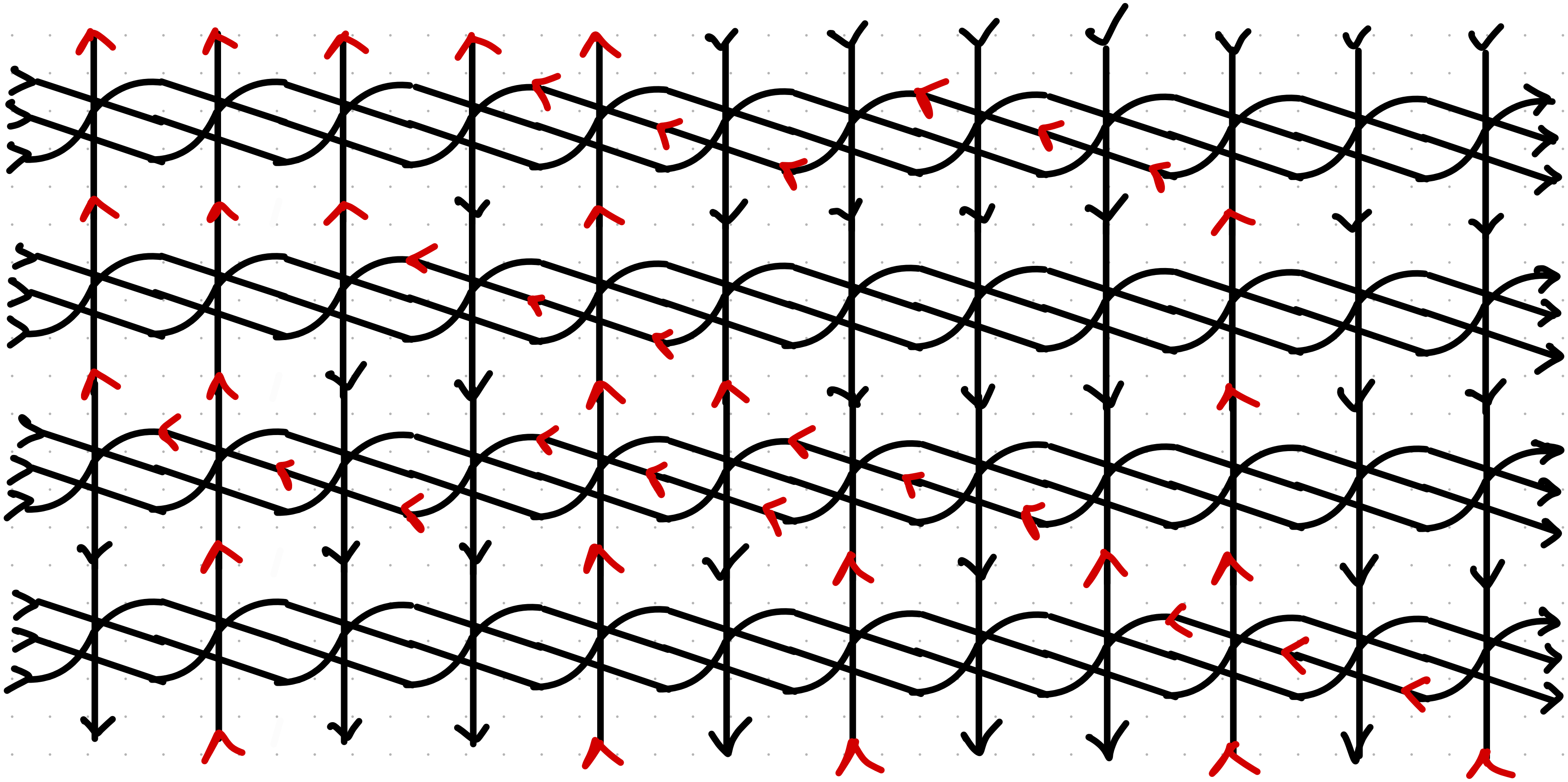
V						
wt(v)	q^s	q^s	0	1	$q^s x_1$	$q^s x_i$

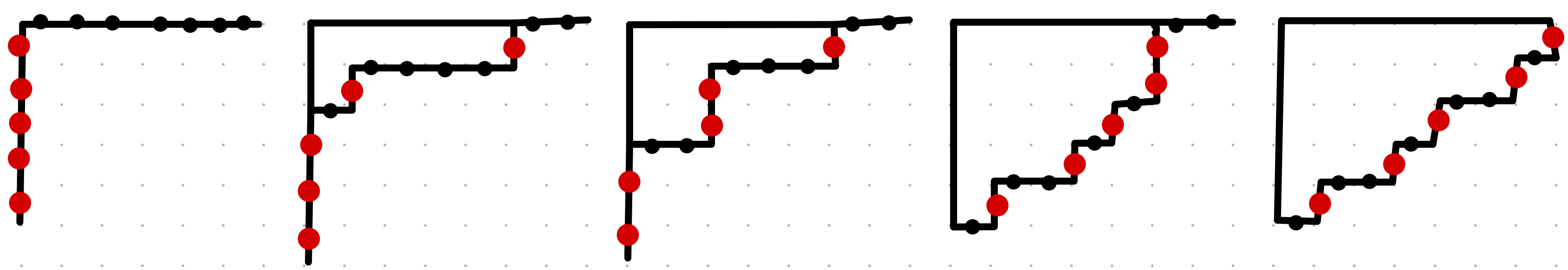
s = # of ◀ in ◯

n-ribbon Lattice Model

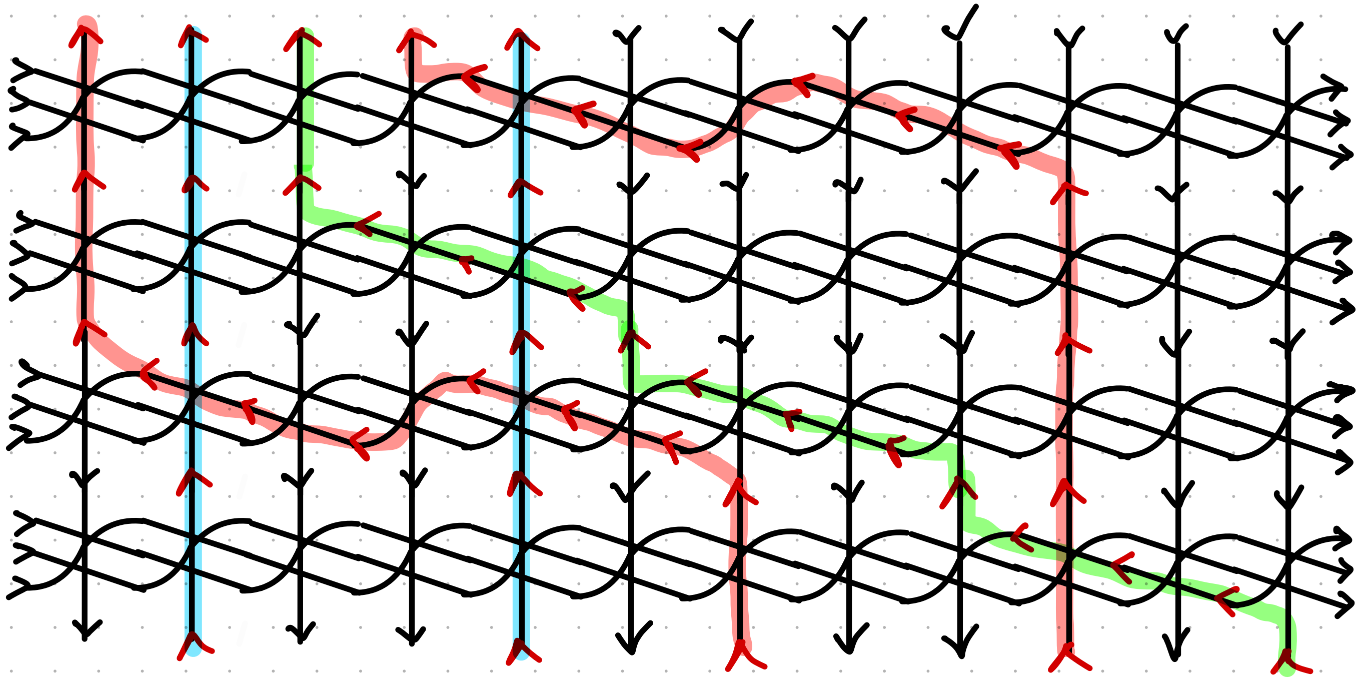


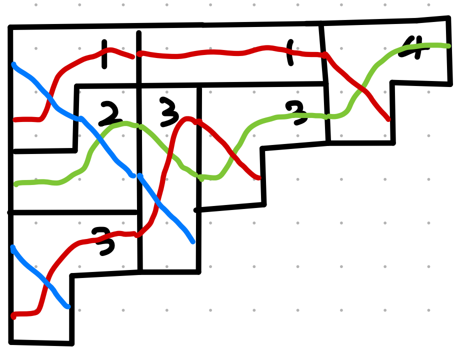
n-ribbon Lattice Model

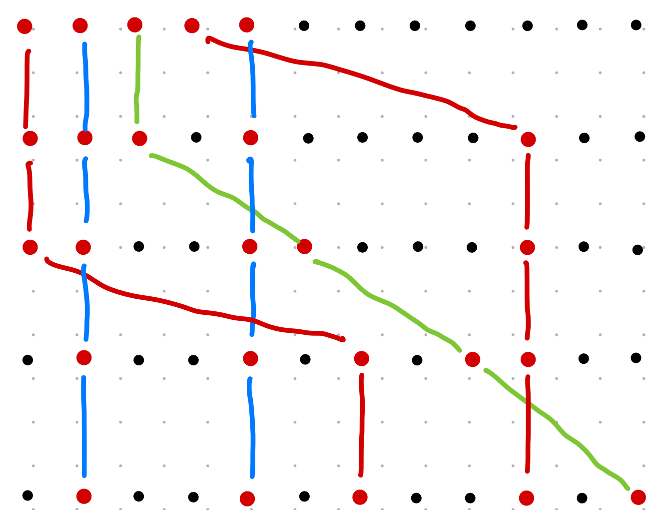




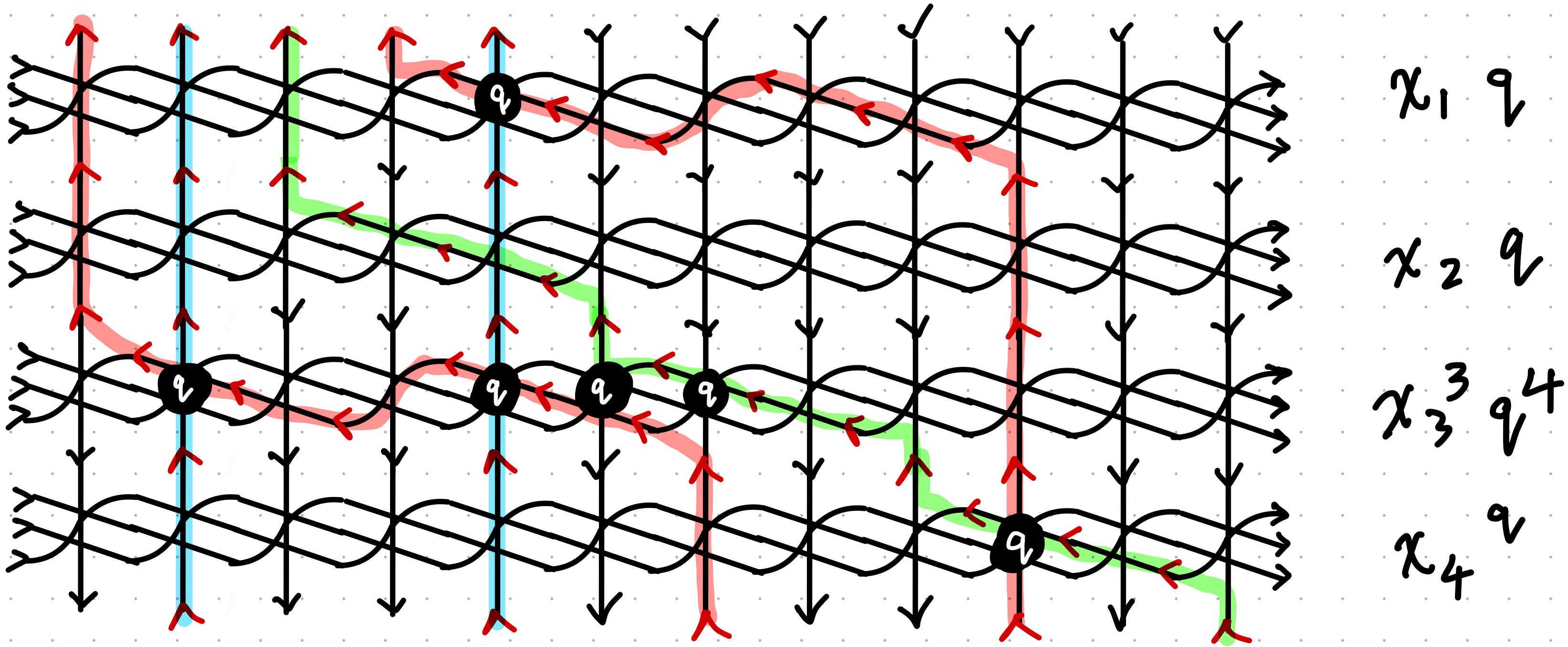
n-Ribbon Lattice = n-colored NILP

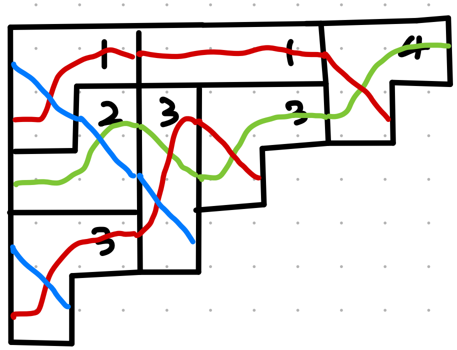


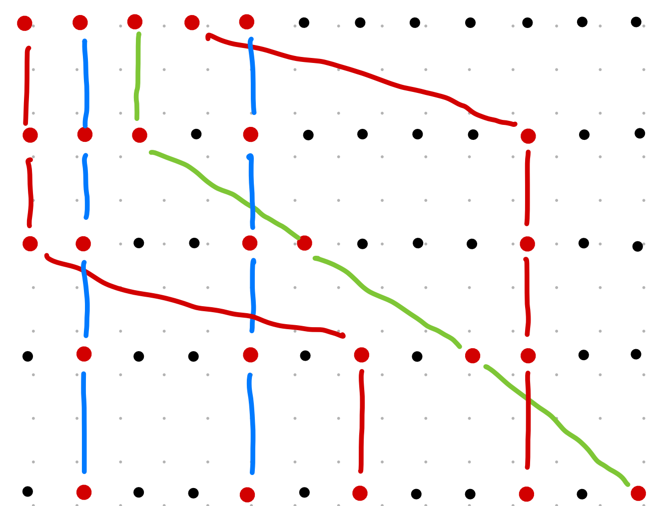




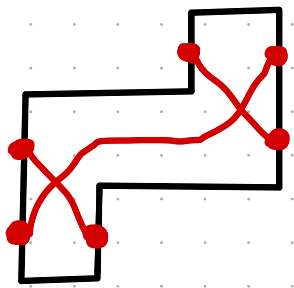
n-Ribbon Lattice = n-colored NILP

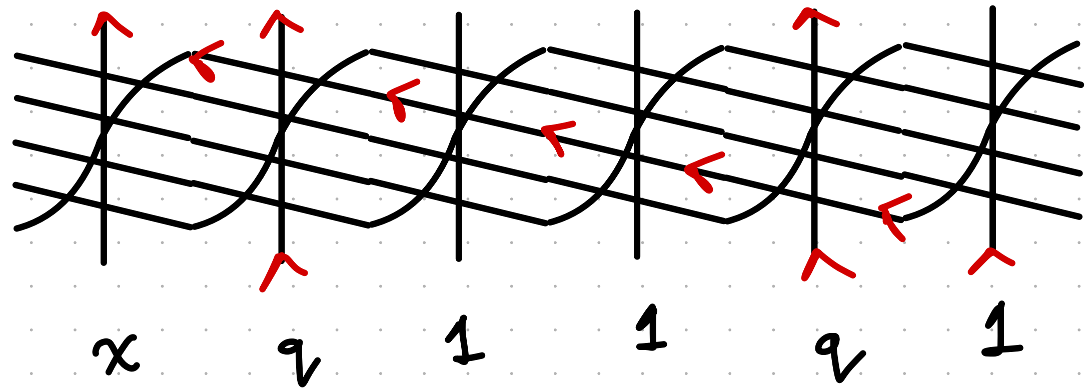






Single Ribbon





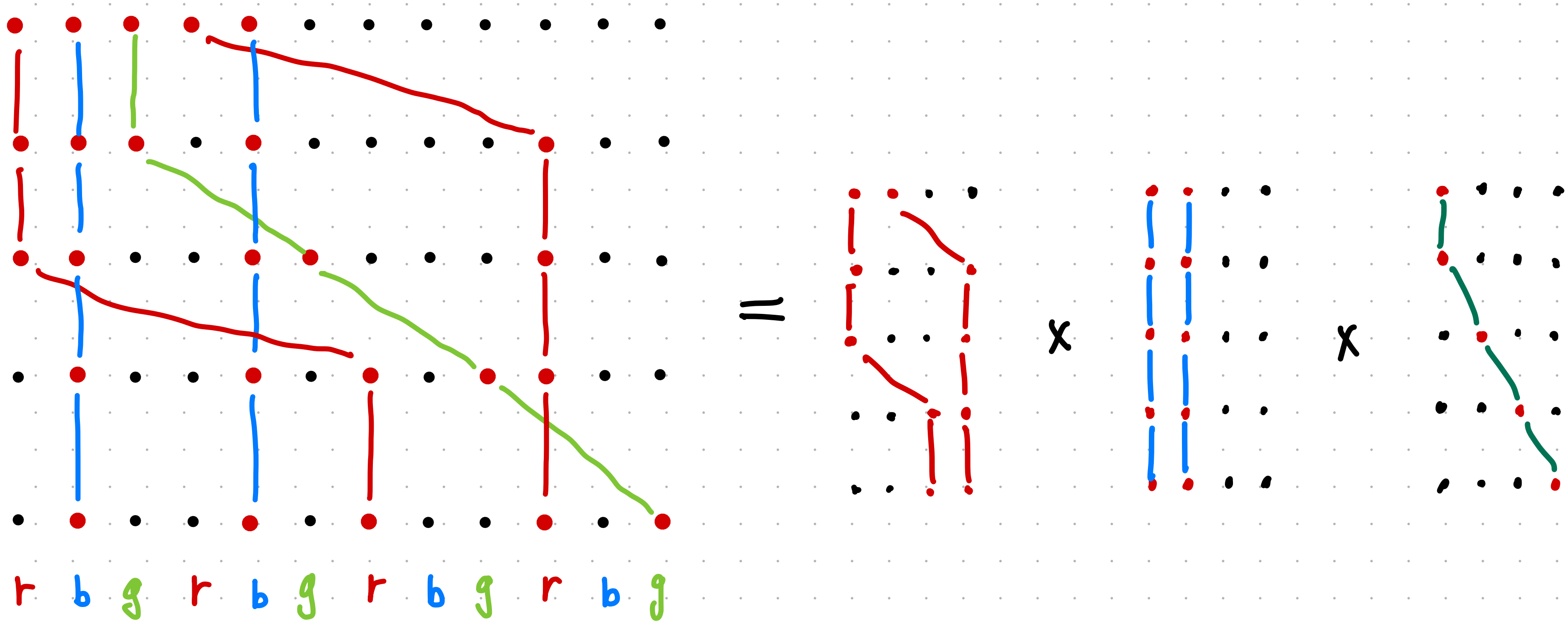
# of intersections is exactly captured by the Boltzmann weights.

Pick up one x-weight at the left-most vertex

Theorem 2 partition function of the ribbon lattice

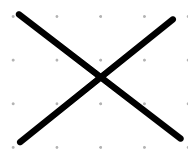
= LLT polynomials !

LLT polynomials are q-analogue of Schur polynomials

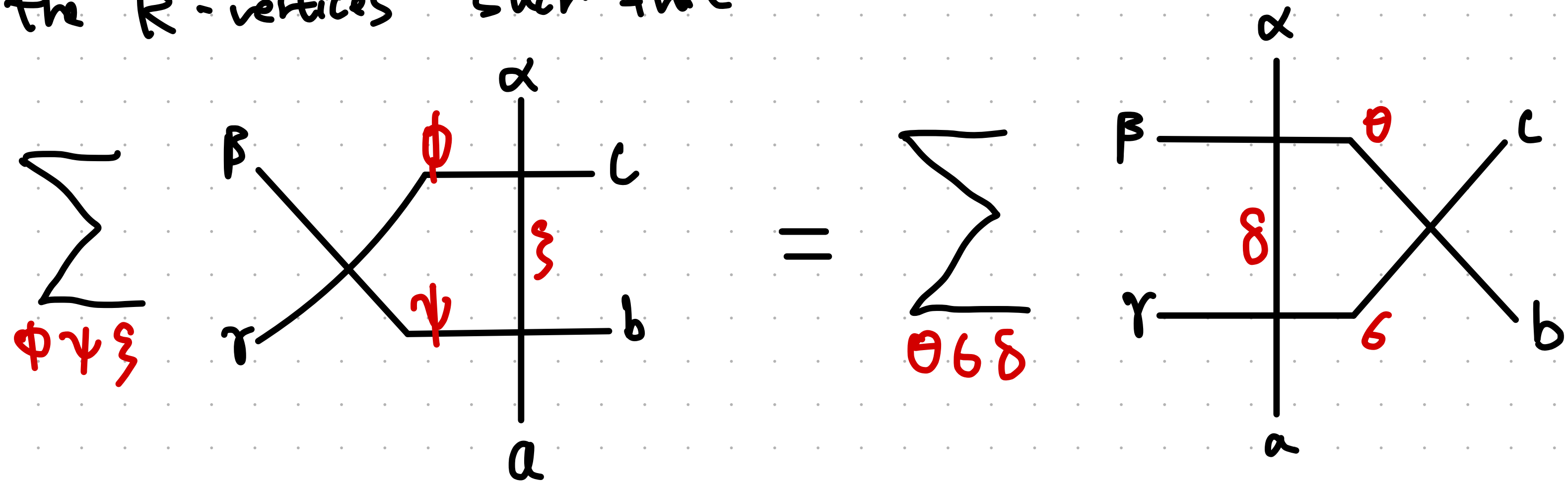


Yang Baxter Equation

For the Ice model, introduce new vertices called $R^{(1)}$ - vertices

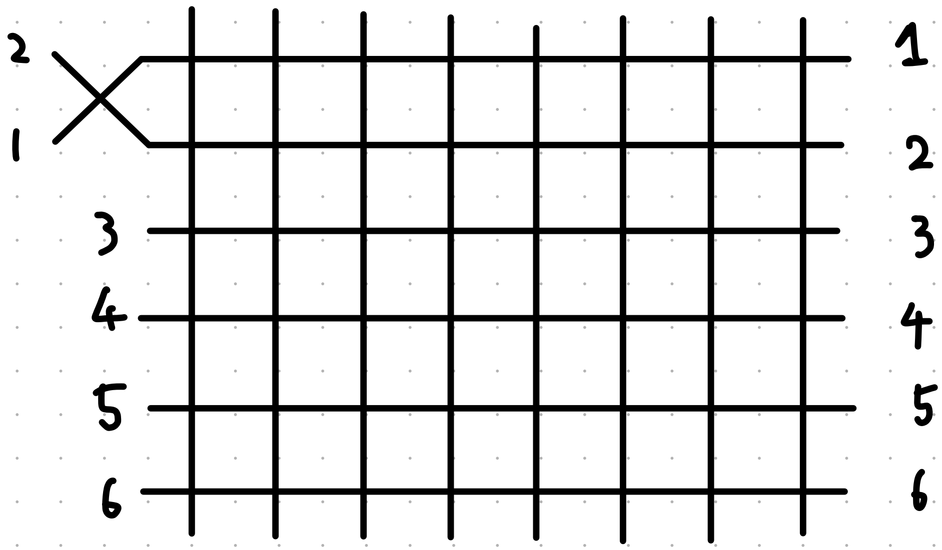


The solution to the Yang Baxter Equation is a set of weights for the $R^{(1)}$ - vertices such that

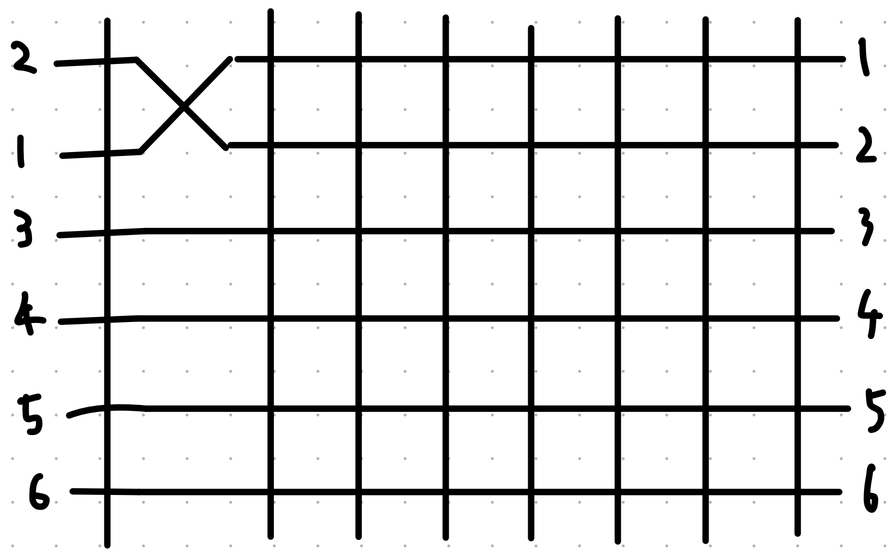


for any boundary arrows $\alpha \beta \gamma a b c$.

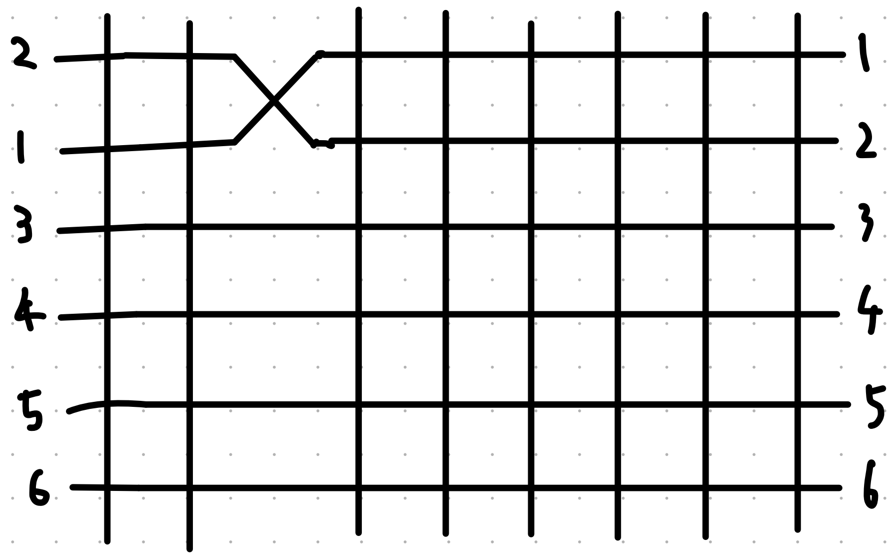
YBE implies Symmetry



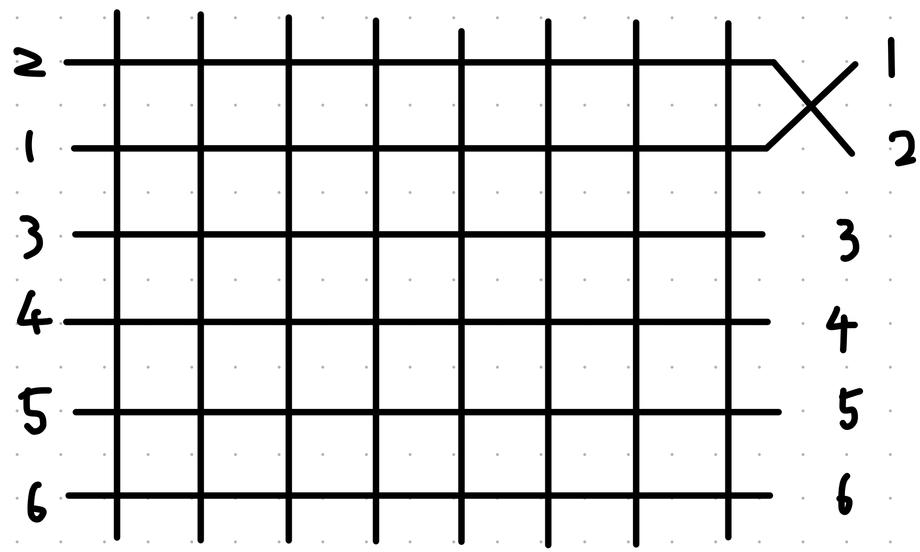
YBE implies Symmetry



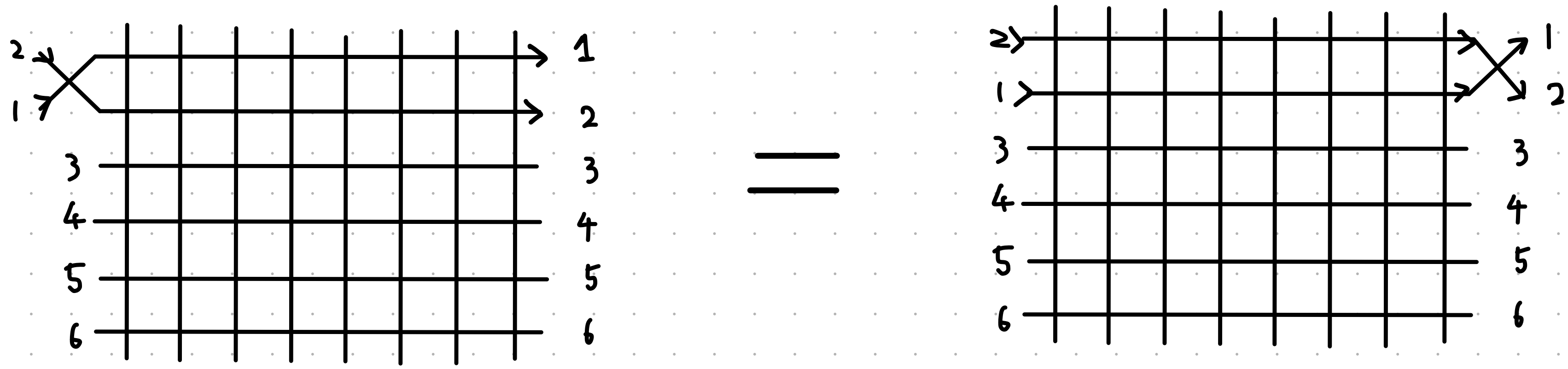
YBE implies Symmetry



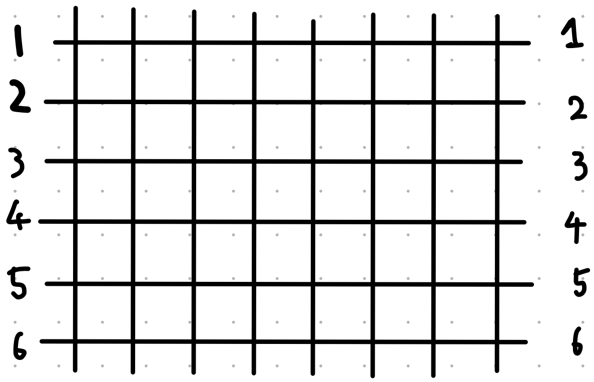
YBE implies Symmetry



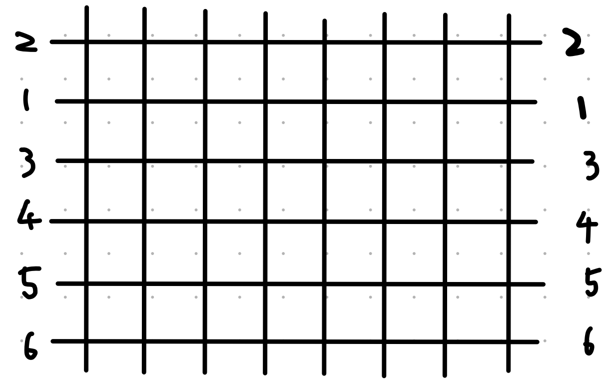
YBE implies Symmetry



YBE implies Symmetry

$\mathrm{wt}(X) \cdot$ 

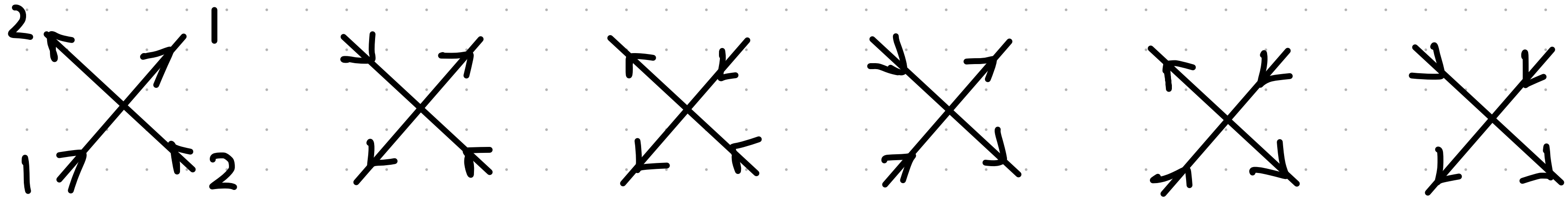
$=$



$\cdot \mathrm{wt}(X)$

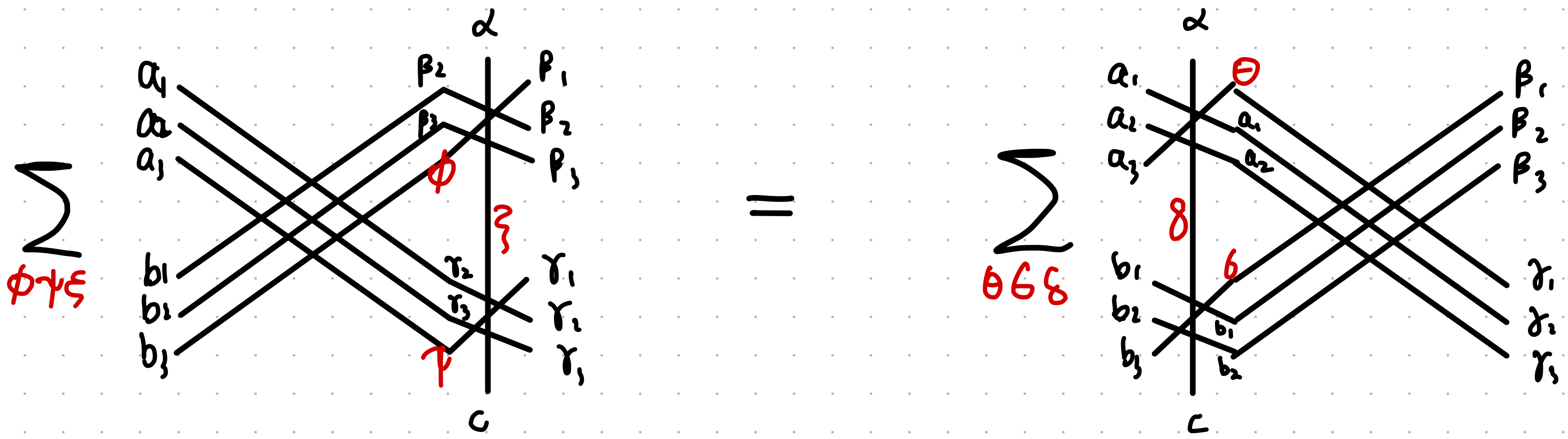
The partition function stays the same but x_1 and x_2's are swapped.

Solution to Schur YBE $R^{(1)}$ weights



$0 \qquad x_2 \qquad x_2 \qquad x_1 \qquad x_1 \qquad x_1 - x_2$

YBE for n-Ribbon $R^{(n)}$-vertices



- Because arrows cannot change on straight edges, most interior edges are fixed (except for 3)
- Therefore it can be solved almost the same way as the 6-vertex model but with some complication from the q's.
- Theorem: The $R^{(n)}$-weights are q-analogue of products of $R^{(1)}$ weights.

Thank You !