# $T$-Path Formula for Decorated Super-Teichmüller Spaces

Sylvester Zhang
(joint work with Gregg Musiker and Nick Ovenhouse)

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School of Mathematics
University of Minnesota

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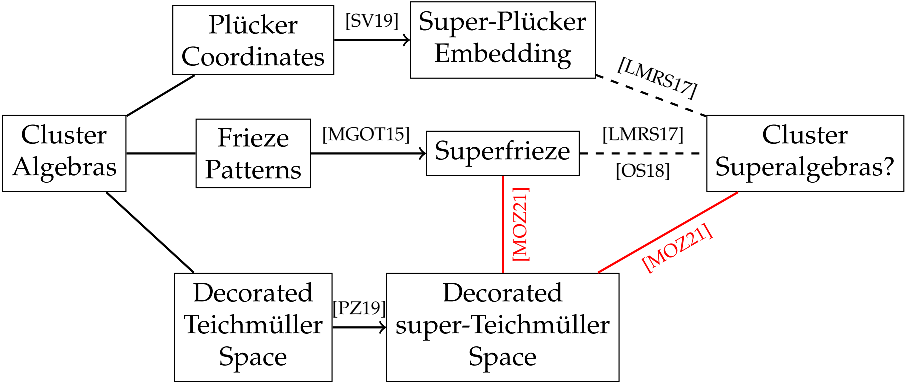
# Motivation




**Goal**

Understand the Cluster structures in decorated super-Teichmüller spaces

# Outline

1. **Motivation**
2. **Cluster Algebras and Decorated Teichmüller Theory**
3. **Decorated Super-Teichmüller Spaces**
4. **Schiffler's $T$-paths**
5. **Main Result: Super $T$-paths**
6. **Super Frieze Patterns and Cluster Superalgebras**

# Outline

# Cluster Algebras

- Cluster algebras, introduced by Fomin and Zelevinsky in 2001, are families of commutative algebras with additional combinatorial structures.
- Elements in a cluster algebra, called *cluster variables*, are grouped into tuples of equal cardinality, called *clusters*. Two clusters with one different entry are linked by a *mutation*.
- A *cluster algebra* is generated from an *initial cluster* along with a mutation rule, often via the help of a *quiver* — a directed graph with no loops and two cycles.

**Definition (Fomin-Zelevinsky 2001)**

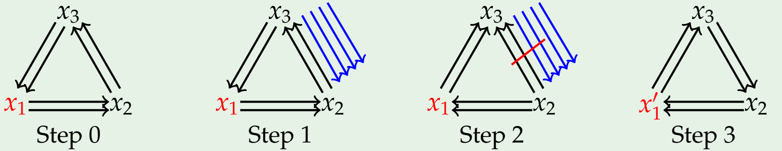
- Fix an integer $n$. A cluster algebra $\mathcal{A}$ is a subring of $\mathbb{R}(x_1, \cdots, x_n)$.
- The ordered tuple $(x_1, \cdots, x_n)$ is called the *initial cluster*.
- A *seed* is a pair $(X, \mathcal{Q})$ where $X$ is a cluster and $\mathcal{Q}$ is a quiver – directed graph with $n$ vertices which corresponds to $n$ elements in a cluster.
- We obtain new clusters from old ones by *mutations*, denoted by $\mu_i$ for $i \in [n]$, as follows
  $$(\{\cdots, x_{i-1}, x_i, x_{i+1}, \cdots\}, \mathcal{Q}) \xrightarrow{\mu_i} (\{\cdots, x_{i-1}, x_i', x_{i+1}, \cdots\}, \mathcal{Q}')$$
  where
  $$x_i x_i' = \prod_{i \to j} x_j + \prod_{j \to i} x_j$$
  The new quiver $\mathcal{Q}'$ is obtained by *quiver mutation*, defined in the next slide.
- Starting from the initial cluster, performing mutations at all possible directions generates the whole cluster algebra.

### Definition (Quiver Mutation)

The new quiver $\mathcal{Q}' = \mu_i(\mathcal{Q})$ is obtain as follows.

1. For every 2-path $j \to i \to k$ in $\mathcal{Q}$, add an arrow $j \to k$,
2. reverse all arrows incident to $i$,
3. remove every new 2-cycles.

### Example




The new cluster variable is: $x_1' = \dfrac{x_2^2 + x_3^2}{x_1}$.

# Decorated Teichmüller Theory

Roughly speaking, the *Teichmüller space* of a surface $F = F_g^s$ is

$$T(F) = \text{the set of hyperbolic structures on F/isotopy.}$$

**Definition**

Consider a smooth oriented surface $F = F_g^s$ with genus $g \geq 0$, punctures $s \geq 0$ and no smooth boundary components. Define the *Teichmüller space* of $F$ to be the quotient space

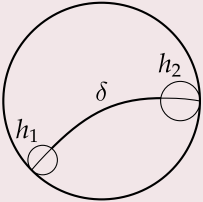
$$T(F) = \mathrm{Hom}\left(\pi_1(F), \mathrm{PSL}(2,\mathbb{R})\right) \big/ \mathrm{PSL}(2,\mathbb{R})$$

**Definition (Penner)**

For any punctured surface $F = F_g^s$ with $s > 0$, the *decorated Teichmüller space* of $F$ is the trivial $\mathbb{R}^s_{>0}$-bundle over $T(F)$, denoted $\tilde{T}(F)$.

The *Poincaré disk*, a model of hyperbolic plane, is defined to be $\mathbb{D} := \{z = x + yi \in \mathbb{C} : |z| < 1\}$, with metric $ds = 2\frac{\sqrt{dx^2+dy^2}}{1-|z|^2}$.

**Definition ($\lambda$-length via horocycles)**




A *horocycle* is a smooth curve in the hyperbolic plane with constant geodesic curvature 1. In $\mathbb{D}$, it's a Euclidean circle tangent to an infinite point, which is the center.
For a pair of horocycles $h_1, h_2$, the *$\lambda$-length* between them is

$$\lambda(h_1, h_2) = e^{\delta/2}$$

where $\delta$ is the hyperbolic distance between the two intersections.

In other words, a decoration is a collection of horocycles above each ideal points.

# Ptolemy Relations

Given a quadruple of horocycles with distinct centers (decorated ideal quadrilateral), one has the Ptolemy transformation (flipping of diagonals).

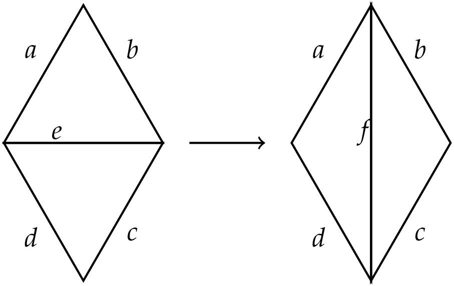



**Figure:** Ptolemy transformation

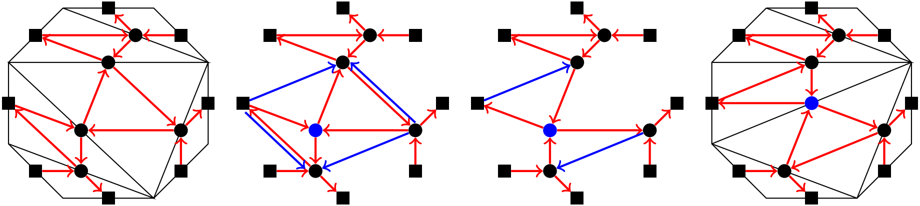
where

$$ef = ac + bd$$

# Ptolemy Relations are Cluster Mutations

Throughout the rest of the paper, let $F$ be a disk with marked points on its boundary (a 'cyclic' polygon).

Associate a quiver to each triangle — draw a vertex for each edge and a triangular quiver  to each triangle. The Ptolemy transformation on $\lambda$-lengths turns out to be the same as cluster mutations.



The exchange relations are exactly the same as Ptolemy relation $ef = ac + bd$, and quiver mutation is the same as flipping diagonals.

# Outline

# Superalgebras

A superalgebra is a $\mathbb{Z}_2$-graded algebra: $A = A_0 \oplus A_1$, with a multiplication $A \times A \to A$ such that $A_i A_j \subset A_{ij}$ $(i, j \in \mathbb{Z}_2)$.

$A_0$ is a commutative algebra itself, which we call the *bosonic* or *even* part of $A$.

$A_1$ is an $A_0$-bimodule, containing elements which anti-commutes. We call it *fermionic* or *odd*.

**Example**

The algebra $A$ generated by $x_1, \cdots, x_n, \theta_1, \cdots, \theta_m$, subject to the following relations

$$x_i x_j = x_j x_i \quad x_i \theta_j = \theta_j x_i \quad \theta_i \theta_j = -\theta_j \theta_i$$

is a superalgebra. Here $A_0$ is spanned by monomials with even number of $\theta$'s and $A_1$ is spanned by monomials with odd number of $\theta$'s.

E.g. $x_1 x_2 + x_1 \theta_1 \theta_3 + x_2 \theta_1 \theta_2 \theta_3 \theta_4 \in A_0$, $x_1 \theta_1 \theta_2 \theta_3 + x_1 x_4 \theta_2 \in A_1$

# Decorated Super-Teichmüller Spaces

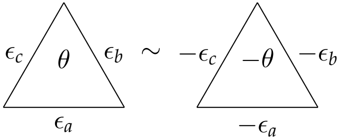
- By replacing $\mathrm{PSL}(2,\mathbb{R})$ with $\mathrm{OSp}(1|2)$, we define the super-Teichmüller space of a surface $F$ to be
$$ST(F) = \mathrm{Hom}(\pi_1(F), \mathrm{OSp}(1|2))/\,\mathrm{OSp}(1|2)$$
- Similar to the bosonic case, the decorated space is encoded by a collection of horocycles centered at each ideal points, which leads to the definition of *super $\lambda$-length*.
- But unlike the bosonic case, we need additional invariants to accommodate for the extra degree of freedom coming from the odd dimension.
- Associate an odd variable to each triangle (triple of ideal points), called the $\mu$-invariants.

# Spin Structures

Components of $ST(F)$ are indexed by the set of *spin structures* on $F$.

Cimasoni-Reshetikhin formulated the set of spin structures of $F$ in terms of the set of isomorphism classes of Kasteleyn orientations of a fatgraph spine of $F$.

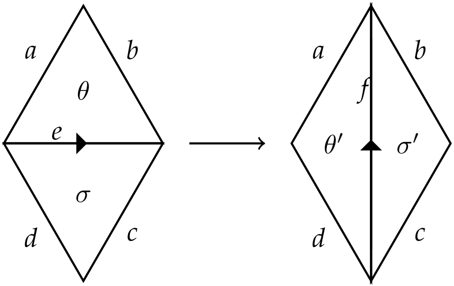
Dual to this formulation, we consider the set of spin structures on $F$ to be the set of equivalence classes of orientations on triangulations of $F$ of the following equivalence relation.




where $\epsilon_a, \epsilon_b, \epsilon_c$ are orientations on the edges, and $\theta$ is the $\mu$-invariant associated to the triangle.

# Super Ptolemy Relation

The Ptolemy transformation on super $\lambda$-length coordinates is given as follows.



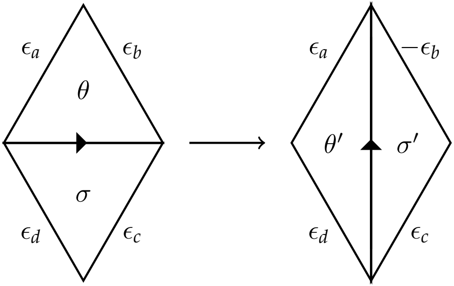

$$ef = ac + bd + \sqrt{abcd}\,\sigma\theta$$

$$\sigma' = \frac{\sigma\sqrt{bd} - \theta\sqrt{ac}}{\sqrt{ac+bd}} \quad \text{and} \quad \theta' = \frac{\theta\sqrt{bd} + \sigma\sqrt{ac}}{\sqrt{ac+bd}}$$

$$\sigma\theta = \sigma'\theta'$$

# Super Ptolemy Relation

Super-flip reverse the orientation of the edge $b$.

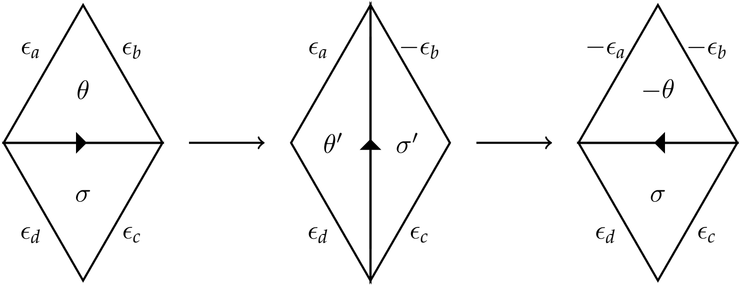



**Remark**

- Super Ptolemy moves are not involution: $\mu_i^8 = I$.
- The odd-degree-0 terms of a super $\lambda$-length are exactly the (ordinary) $\lambda$-length in the bosonic decorated space.

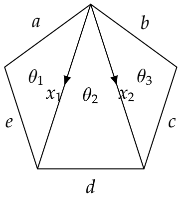
# Super Ptolemy Relation

If we flip a diagonal twice:




The orientations of the triangle $\theta$ are reversed and $\theta$ is changed to $-\theta$.

# Super Ptolemy Relation - Example



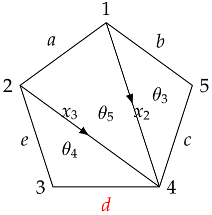

Start with a Pentagon with given orientation.

The boundary orientations are ignored, because they are irrelevant in the calculations.

What is $\lambda_{2,3}$?

We first flip the edge $x_1$.

# Super Ptolemy Relation - Example




After flipping $x_1$ to $x_3$, we get:

$$x_3 = \frac{ad + ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2$$

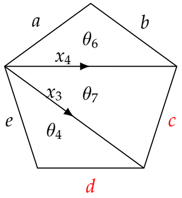
$$\theta_4 = \frac{\sqrt{ad}\,\theta_1 - \sqrt{ex_2}\,\theta_2}{\sqrt{x_1x_3}}$$

$$\theta_5 = \frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1x_3}}$$

Here the red color indicates that the orientation has been reversed.

Next we flip $x_2$.

# Super Ptolemy Relation - Example




After flipping $x_2$ to $x_4$, we have:

$$x_4 = \frac{ac + bx_3}{x_2} + \frac{\sqrt{acbx_3}}{x_2}\,\theta_5\theta_3$$

$$= \frac{acx_1 + abd + bex_2}{x_1x_2} + \frac{b\sqrt{adex_2}}{x_1x_2}\theta_1\theta_2 +$$

$$\frac{\sqrt{acb\left(\frac{ad+ex_2}{x_1} + \frac{\sqrt{adex_2}}{x_1}\theta_1\theta_2\right)}}{x_2}\left(\frac{\sqrt{ad}\,\theta_2 + \sqrt{ex_2}\,\theta_1}{\sqrt{x_1x_3}}\right)\theta_3$$

$$= \frac{acx_1}{x_1x_2} + \frac{abd}{x_1x_2} + \frac{bex_2}{x_1x_2} + \frac{b\sqrt{ade}}{x_1\sqrt{x_2}}\theta_1\theta_2 +$$

$$\frac{a\sqrt{bcd}}{\sqrt{x_1}x_2}\theta_2\theta_3 + \frac{\sqrt{abcd}}{\sqrt{x_1x_2}}\theta_1\theta_3$$

# Main Question

In a cluster algebra $A$, any cluster variable can be expressed as a positive Laurent polynomial in the initial cluster, i.e.

$$A \subset \mathbb{R}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}].$$

### Questions

- Does the super $\lambda$-length satisfy some Laurent phenomenon?
- Is there a "positivity" for terms with anti-commuting variables?

### Answers (Spoiler Alert)

- Super $\lambda$-lengths live in $\mathbb{R}[x_1^{\pm\frac{1}{2}}, \cdots, x_1^{\pm\frac{1}{2}} | \theta_1, \cdots, \theta_{n+1}]$.
- There exists an ordering on the odd variables, called *positive ordering*, such that if we multiply $\theta$'s in the positive ordering then the coefficients are positive.

# Outline

# Schiffler's $T$-paths

Consider the graph $T$ coming from a triangulated polygon.

A *T-path* from $i$ to $j$ is a path in $T$ starting at vertex $i$, ending at $j$, such that

**(T1)** the path does not use any edge twice

**(T2)** the path has an odd number of edges

**(T3)** the even-numbered edges cross the diagonal $(i, j)$

**(T4)** The intersections of the path and $(i, j)$ move from progressively $i$ to $j$.

Let $T_{ij}$ denote the set of $T$-paths from $i$ to $j$.

For a $T$-path $\gamma = (x_1, x_2, \cdots)$, define it's weight to be

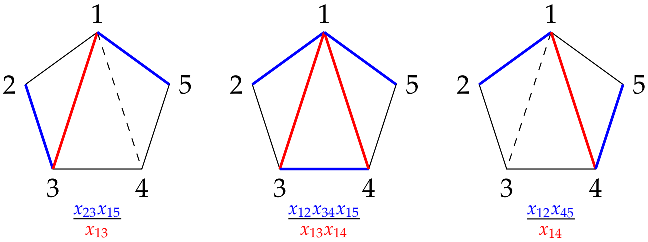
$$\mathrm{wt}(\gamma) = \prod_{i \text{ odd}} \lambda(x_i) \prod_{i \text{ even}} \lambda(x_i)^{-1}$$

where $\lambda(x_i)$ denote the $\lambda$-length of the edge $x_i$.

**Theorem (Schiffler)**

$$\lambda(x_{i,j}) = \sum_{t \in T_{i,j}} \mathrm{wt}(t)$$

Here are all the $T$-paths in $T_{25}$. (odd steps are blue and even steps are red)



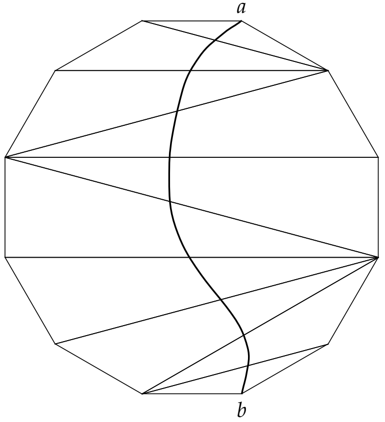

$$\lambda(x_{2,5}) = \sum_{t \in T_{25}} \mathrm{wt}(t) = \frac{x_{23}x_{15}}{x_{13}} + \frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}}$$

# Outline

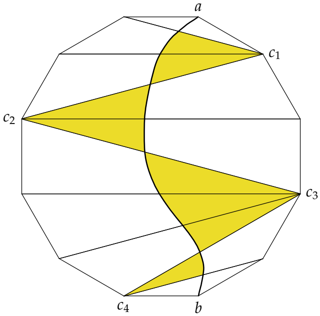
# Super $T$-paths

From now on we only consider triangulations with a longest arc crossing all internal diagonals.

In other words, every triangle has a boundary edge. Call the end points of the longest arc $a$ and $b$.



# Fan Decomposition



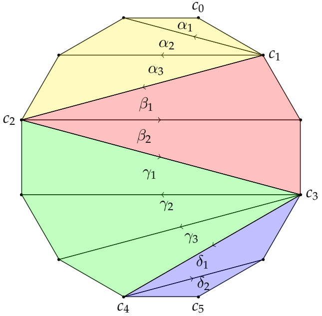

For a triangulation $T$, we will define a canonical *fan decomposition*.

The arc $(a, b)$ intersect with internal diagonals, and create smaller triangles (colored yellow).

Vertices of these yellow triangles are called *fan centers*, denoted $c_1, \cdots, c_n$, ordered by their distance from $a$. And we further denote $a = c_0$ and $b = c_{n+1}$.

The sub-triangulation bounded by $c_{i-1}, c_i, c_{i+1}$ is called the $i$-th fan segment of $T$.

# Default Orientation and Positive Ordering




Define a *default orientation* on the interior diagonals.

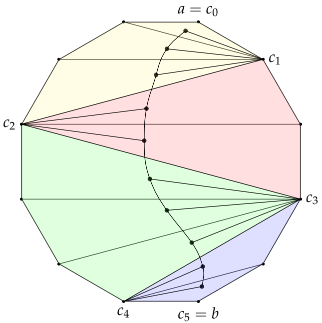
- Edges inside each fan segments are directed away from the center.
- Others are oriented as $c_1 \to c_2 \to \cdots \to c_n$.

Define a *positive ordering* on $\mu$-invariants.

- $\mu$-invariants in a fan are ordered counterclockwise around the center.
- "Alternate" across the fans.

$$\alpha_1 > \alpha_2 > \alpha_3 > \gamma_1 > \gamma_2 > \gamma_3 > \delta_2 > \delta_1 > \beta_2 > \beta_1$$

# The Auxiliary Graph




For each triangle in $T$, we place an *internal vertex*.

The internal vertices are connected to the nearest fan centers by $\sigma$-edges. The $\sigma$-edges are considered to cross the arc $(a, b)$.

Every pair of internal vertices are connected by a teleportation, called a $\tau$-edge. (Note that the $\tau$-edges are drawn to be overlapping.)

The resulting graph $\Gamma_T^{a,b}$ is the *auxiliary graph* associated to $\{T, a, b\}$.

# Super $T$-paths

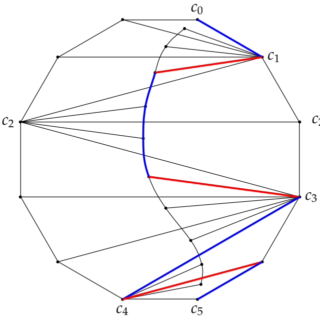
Finally, we define super $T$-paths to be paths on the auxiliary graph such that:

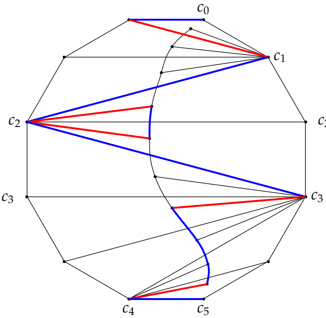
- **(T1)** the path does not use any edge twice.
- **(T2)** the path has an odd number of edges.
- **(T3)** the even-numbered edges cross the diagonal $(a, b)$.
- **(T4)** The intersections of the path and $(a, b)$ move from progressively $a$ to $b$.
- **(T5)** $\sigma$-edges must be even and $\tau$-edges must be odd.

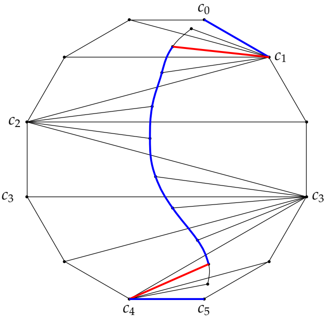
Let $\tilde{T}_{a,b}$ denote the set of super $T$-path on $\Gamma_T^{a,b}$.

Note that, every ordinary $T$-path is also a super $T$-path: $T_{a,b} \subset \tilde{T}_{a,b}$

# Super $T$-paths: Examples



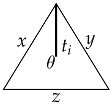



# Weights of Super $T$-paths

If a super $T$-path uses edges $t_1, t_2, \ldots,$ we define its weight as follows.

- If $t_i$ is a diagonal in the triangulation, then:
  $\mathrm{wt}(t_i) = \lambda(t_i)$ if $i$ odd, and
  $\mathrm{wt}(t_i) = \lambda(t_i)^{-1}$ if $t$ is even.
- If $t_i$ is a $\tau$-edge, then $\mathrm{wt}(t_i) = 1$
- If $t_i$ is a $\sigma$-edge, then $\mathrm{wt}(t_i) = \tilde{\theta} := \sqrt{\frac{z}{xy}}\,\theta$. Here $x, y, z$ are $\lambda$-lengths and $\theta$ is the $\mu$-invariant.




If $t$ is a super $T$-path with edges $t_1, t_2, \ldots,$ define $\mathrm{wt}(t) = \prod_i \mathrm{wt}(t_i)$.
Here the product is take under the positive ordering.

# Main Theorem

**Theorem (Musiker-Ovenhouse-Z. 2021)**

*Under default orientation, the super $\lambda$-length of the arc $(a, b)$ (assuming to be the longest arc in $T$) is given by:*

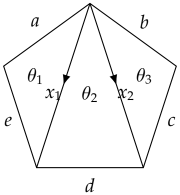
$$\lambda(a, b) = \sum_{t \in \tilde{T}_{a,b}} \mathrm{wt}(t)$$

With the following lemma, we can apply the main theorem for triangulations with arbitrary orientation.

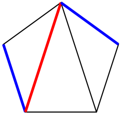
**Lemma (Musiker-Ovenhouse-Z. 2021)**

*In the equivalence class of any spin structure, there exists (at least) a default orientation.*

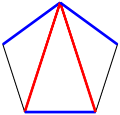
# Formula for $\lambda$-lengths: Example



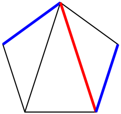

$\theta_1 > \theta_2 > \theta_3$



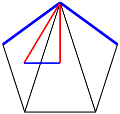
$\frac{be}{x_1}$



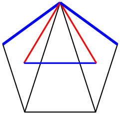
$\frac{abd}{x_1x_2}$



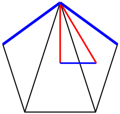
$\frac{ac}{x_2}$



$ab\sqrt{\frac{e}{ax_1}\theta_1}\sqrt{\frac{d}{x_1x_2}\theta_2}$



$ab\sqrt{\frac{e}{ax_1}\theta_1}\sqrt{\frac{c}{bx_2}\theta_3}$



$ab\sqrt{\frac{d}{x_1x_2}\theta_2}\sqrt{\frac{c}{bx_2}\theta_3}$

# Formula for $\mu$-invariants

**Theorem (Musiker-Ovenhouse-Z. 2021)**

*Let $T$ be a triangulation with $a = c_0, c_1, \cdots, c_{n+1} = b$ its fan centers. Let $\Theta$ denote the set of all internal vertices in $\Gamma_T^{a,b}$. Then*

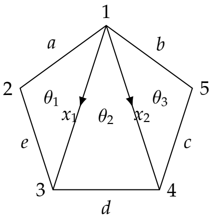
$$\sqrt{\frac{\lambda(a,b)\lambda(b,c_1)}{\lambda(a,c_1)}}\boxed{abc_1} = \sum_{\theta\in\Theta} \mathrm{wt}\{\text{'partial' super } T\text{-path from } a \text{ to } \theta\}$$

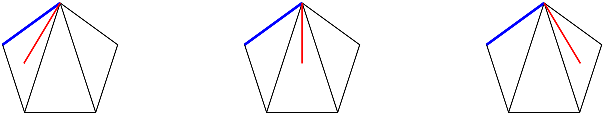
*Here* wt *means the weighted sum, and a partial super T-path satisfies all axioms except having even number of edges.*

**Remark**

Note that the above theorem only covers a special family of triangles. The $\mu$-invariants themselves don't have simple expansions, because the $\lambda$-lengths in the term $\sqrt{\frac{\lambda(a,b)\lambda(b,c_1)}{\lambda(a,c_1)}}$ are not always in the triangulation.

# Formula for $\mu$-invariants: Example



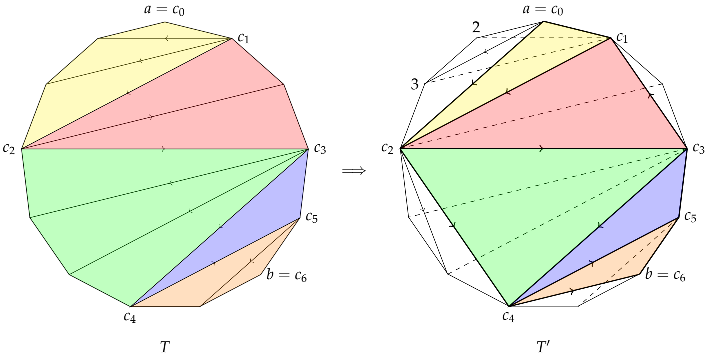



$$a\sqrt{\frac{e}{ax_1}}\theta_1 \qquad a\sqrt{\frac{d}{x_1x_2}}\theta_2 \qquad a\sqrt{\frac{c}{bx_2}}\theta_3$$

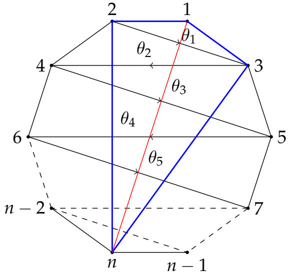
$$\sqrt{\frac{b\lambda_{25}}{a}}\boxed{125} = \sqrt{\frac{ae}{x_1}}\theta_1 + \frac{a\sqrt{d}}{\sqrt{x_1x_2}}\theta_2 + \frac{a\sqrt{c}}{\sqrt{bx_2}}\theta_3$$

# Proof Sketch - Three Steps

- We first prove our Theorems for single-fan triangulations.
- Next we prove in the case of zig-zag triangulations.
- Finally we prove in full generality by combining the above two cases using the following sequence of flips.



# Proof Sketch - Double Helix Induction



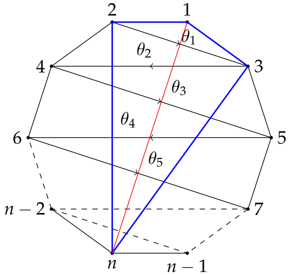

$$\boxed{12n}\sqrt{\frac{\lambda_{1n}\lambda_{2n}}{\lambda_{12}}} = \underbrace{\boxed{23n}\sqrt{\frac{\lambda_{3n}\lambda_{2n}}{\lambda_{23}}}}_{\text{1st term}} + \underbrace{\boxed{123}\sqrt{\frac{\lambda_{13}}{\lambda_{12}\lambda_{23}}}\lambda_{2n}}_{\text{2nd term}}$$

**1st term:** (by induction hypothesis) all partial super $T$-path starting from $n$ and ending at one of $\theta_2, \theta_3, \cdots$.

**2nd term:** all complete super $T$-path from $n$ to 2 plus an $\sigma$-edge to $\theta_1$.

**1st + 2nd:** partial super $T$-paths from $n$ to one of $\theta_1, \theta_2, \theta_3, \cdots$.

# Proof Sketch - Double Helix Induction




$$\lambda_{1n} = \underbrace{\frac{\lambda_{12}\lambda_{3n}}{\lambda_{23}}}_{\text{part 1}} + \underbrace{\frac{\lambda_{13}\lambda_{2n}}{\lambda_{23}}}_{\text{part 2}} - \underbrace{\sqrt{\frac{\lambda_{12}\lambda_{13}}{\lambda_{23}}}\boxed{123} \cdot \sqrt{\frac{\lambda_{2n}\lambda_{3n}}{\lambda_{23}}}\boxed{23n}}_{\text{part 3}}$$

**part 1:** $\tilde{T}_{1,n}$ whose first two steps are $(1,2)$ and $(2,3)$.

**part 2:** $\tilde{T}_{1,n}$ whose first step is $(1,3)$.

**part 1+2:** $\tilde{T}_{1,n}$ with out using $\theta_1 = \boxed{123}$.

**part 3:** By induction hypothesis of $\boxed{23n}$, part 3 has all super $T$-path from 1 to $n$ which used $\theta_1$.

**part 1+2+3:** Together gives all super $T$-path from 1 to $n$.

# Outline

# Superfriezes

Supersymmetric frieze patterns are introduced by Morier-Genoud, Ovsienko, and Tabachnikov. They are the following array of numbers

$$
\begin{array}{cccccccccccc}
 & \dots & 0 & & & & 0 & & & & 0 & \\
\dots & 0 & & 0 & & 0 & & 0 & & 0 & & \dots \\
1 & & & & 1 & & & & 1 & & & \dots \\
 & \varphi_{0,0} & & \varphi_{\frac{1}{2},\frac{1}{2}} & & \varphi_{1,1} & & \varphi_{\frac{3}{2},\frac{3}{2}} & & \varphi_{2,2} & & \dots \\
 & & f_{0,0} & & & & f_{1,1} & & & & f_{2,2} & \\
 & \varphi_{-\frac{1}{2},\frac{1}{2}} & & \varphi_{0,1} & & \varphi_{\frac{1}{2},\frac{3}{2}} & & \varphi_{1,2} & & \varphi_{\frac{3}{2},\frac{5}{2}} & & \dots \\
f_{-1,0} & & & & f_{0,1} & & & & f_{1,2} & & & \\
 & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots \\
 & & f_{2-m,1} & & & & f_{0,m-1} & & & & f_{1,m} & \\
\dots & \varphi_{\frac{3}{2}-m,\frac{3}{2}} & & \varphi_{2-m,2} & & \dots & & \varphi_{0,m} & & \varphi_{\frac{1}{2},m+\frac{1}{2}} & & \varphi_{1,m+1} \\
1 & & & & 1 & & & & 1 & & & \\
\dots & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
 & \dots & 0 & & & & 0 & & & & 0 & \dots
\end{array}
$$

# Super Diamond

A super frieze is built up by *super diamonds* as follows.

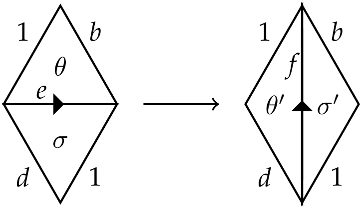
$$\begin{array}{ccccc} & & B & & \\ & \Xi & & \Psi & \\ A & & & & D \\ & \Phi & & \Sigma & \\ & & C & & \end{array}$$

Every super diamond is a matrix in $\mathrm{OSp}(1|2)$, satisfying the following frieze rules:

$$\begin{aligned} AD - BC &= 1 + \Sigma\Xi \\ A\Sigma - C\Xi &= \Phi \\ B\Sigma - D\Xi &= \Psi \\ B\Phi - A\Psi &= \Xi \\ D\Phi - C\Psi &= \Sigma \\ \Sigma\Xi &= \Psi\Phi \end{aligned}$$

# Super Diamonds as Ptolemy Relations

Consider quadrilateral flip as follows where two of the edges have length 1.




The Ptolemy relation is equivalent to the superfrieze relation of the following diamond:

$$
\begin{array}{ccccc}
 & & b & & \\
 & \theta\sqrt{be} & & \sigma'\sqrt{bf} & \\
e & & & & f \\
 & \sigma\sqrt{ed} & & \theta'\sqrt{df} & \\
 & & d & &
\end{array}
$$

Set $\tilde{\theta} = \theta\sqrt{be}$, $\tilde{\sigma} = \sigma\sqrt{ed}$, $\tilde{\theta}' = \theta'\sqrt{df}$, and $\tilde{\sigma}' = \sigma'\sqrt{bf}$.

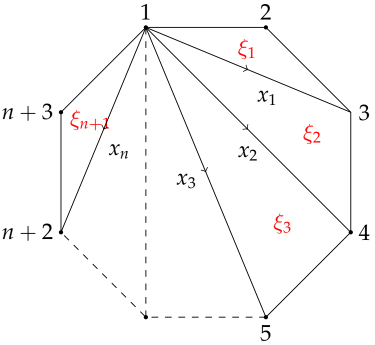
# Superfriezes from a marked disk

As a corollary of the previous slide, we have

**Theorem (Musiker-Ovenhouse-Z. 2021)**

*Every (finite) superfrieze pattern come from the super $\lambda$-lengths and $\mu$-invariants of a marked disk.*

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
 & {\color{red}\xi_1} & & & & & & & \\
 & & x_1 & & & & & & \\
 & & & {\color{red}\xi_2} & & & & & \\
 & & & & x_2 & & & & \\
 & & & & & \ddots & & & \\
 & & & & & & x_n & & \\
 & & & & & & & {\color{red}\xi_{n+1}} & \\
 & & & & & & & & 1
\end{array}$$



# Relation to Ovsienko-Shapiro Cluster Algebra

Ovsienko and Shapiro [OS18] proposed a Cluster superalgebra using *extended quivers*.

For every super diamond, associate an extended quiver:

$$\begin{array}{ccccc} & & b & & \\ & \tilde{\theta} & & \tilde{\sigma}' & \\ e & & & & f \\ & \tilde{\sigma} & & \tilde{\theta}' & \\ & & d & & \end{array}$$

$$\begin{tikzcd} & {\color{red}\tilde{\theta}} \arrow[dr] & & {\color{red}\tilde{\theta}'} & \\ b & & e \arrow[ll] \arrow[ur] \arrow[rr] & & d \end{tikzcd}$$

Note that $\tilde{\theta}$ and $\tilde{\theta}'$ are not in the same triangulation!

**Question**

Can we add odd mutations $\tilde{\sigma} \to \tilde{\theta}'$ and $\tilde{\theta} \to \tilde{\sigma}'$, turning the extended quiver mutation into Ptolemy transformation?

# Thank you!

Special thanks to Nick for sharing the LaTeX source code from his MSU talk!

# Thank You!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