1.5.3  (a) Any union of open sets is open. Look at any set of open sets \( \{ A_\alpha \} \). If \( x \in \bigcup \alpha A_\alpha \), then by definition of union, \( x \in A_\alpha \) for some particular \( \alpha \). Since \( A_\alpha \) is open, there is some \( r > 0 \) so that \( B_r(x) \subset A_\alpha \). Then (again by the definition of union) \( B_r(x) \subset \bigcup \alpha A_\alpha \).

Does this work for infinitely many open sets? Yes – the same argument works.

(b) \( \bigcap_{i=1}^n A_i \) is open.

If we have two open sets \( A_1 \) and \( A_2 \), their intersection is open: If the intersection is empty, it’s “trivially open” (the empty set is open). If the intersection is not empty, there’s some \( x \in A_1 \cap A_2 \). Since \( A_1, A_2 \) are open, there are positive \( r_1 \) and \( r_2 \) so that \( B_{r_1}(x) \subset A_1 \) and \( B_{r_2}(x) \subset A_2 \). We can choose \( r = \min(r_1, r_2) \) and be confident that \( B_r(x) \subset A_1 \cap A_2 \).

If we have \( n \) open sets, instead, we can repeat the argument, as most of you did... but almost everyone missed a crucial point that I’ll put in bold! If the intersection of \( n \) sets is empty, then we’re done. If the intersection is not empty, then by definition of intersection and non-emptiness there’s some \( x \in A_i \) for all \( A_i \). Since each \( A_i \) is open, we have \( B_{r_i}(x) \subset A_i \) for some \( r_i > 0 \), for each \( i \). We can choose \( r = \min(r_1, r_2, \ldots, r_n) \) because there are only a finite number of \( r_i \)s to look at. This is an important point. After all, look at part (c)! Why doesn’t the proof of (b) apply to the example you give in (c)?

(c) Given an example to show that \( \bigcap_{n=1}^\infty A_i \) need not be open.

A classic example is \( \bigcap_{n=1}^\infty \left( -\frac{1}{n}, \frac{1}{n} \right) \). Notice how this is written out: it is the intersection of an infinite number of intervals. A number of people had some trouble expressing this. (Look at your notation and ask yourself if you said what you meant, and ponder the meaning of an infinite intersection of points...)

1.5.10  Is it true that for matrices \( A, B \) we have \( e^{A+B} = e^A e^B \)?

No: look at the power series expansions.

\[
e^{A+B} = \sum_{n=0}^\infty \frac{1}{n!} (A + B)^n
\]

\[
= 1 + (A + B) + \frac{1}{2} (A + B)^2 + \cdots
\]

\[
= 1 + A + B + \frac{1}{2} A^2 + \frac{1}{2} AB + \frac{1}{2} BA + \frac{1}{2} B^2 + \cdots
\]

while
\[ e^A e^B = \left( \sum_{i=0}^{\infty} \frac{1}{i!} A^n \right) \left( \sum_{i=0}^{\infty} \frac{1}{i!} B^n \right) \]
\[ \quad = (1 + A + \frac{1}{2} A^2 + \cdots)(1 + B + \frac{1}{2} B^2 + \cdots) \]
\[ \quad = (1 + AB + \frac{1}{2} A^2 + \frac{1}{2} B^2 + \cdots) \]

We can compare just the degree two terms: in both expansions we have \( \frac{1}{2} A^2 \) and \( \frac{1}{2} B^2 \), but in the first we have \( \frac{1}{2} AB + \frac{1}{2} BA \) and in the second we have \( AB \). Are these equal? In general, only if \( A \) and \( B \) commute.

(b) We said in part A that if \( A \) and \( B \) commute, we have \( \frac{1}{2} AB + \frac{1}{2} BA = AB \). So do \( A \) and \( A \) commute? Yes. Thus \( (e^A)^2 = e^A e^A = e^{A^2} = e^{2A} \).

1.5.14 Find the following limits:

(a) \[ \lim_{(x,y) \to (1,2)} \frac{x^2 + y}{x^2 + y} \] Since we are dealing with a rational function that does not have vanishing denominator at \((1,2)\), we can just evaluate at this point (it’s a Corollary). Get \( \frac{1}{3} \).

(b) \[ \lim_{(x,y) \to (0,0)} \sqrt{1 - x^2 - y^2} \] Not so easy. Test some cases: If we approach the origin along \( x = y \), have \( \sqrt{\frac{y^2}{2x^2}} = \frac{1}{2\sqrt{|y|}} \), which we can see goes to infinity as \( x \to 0 \). On the other hand, if we let \( x = 0 \) and approach along the \( y \)-axis, we get that the limit is zero (the top of the fraction is zero and that’s that). So the limit of this function does not exist at \((0,0)\).

(c) \[ \lim_{(x,y) \to (0,0)} \frac{\sqrt{xy}}{\sqrt{x^2 + y^2}} \] Also somewhat tricky. Let’s try approaching along the line \( y = mx \). Then we’re looking at \( \lim_{(x,y) \to (0,0)} \frac{\sqrt{mx^2}}{\sqrt{(1+m^2)x^2}} = \lim_{(x,y) \to (0,0)} \frac{\sqrt{m}}{\sqrt{1+m^2}} \), which varies as we vary \( m \). Again the limit of this function does not exist at \((0,0)\).

(d) Trick question! After all that, \( \lim_{(x,y) \to (1,2)} x^2 + y^3 - 3 \) is easy – we’re looking at a polynomial, so we can simply evaluate to get \( 1 + 8 - 3 = 6 \).

1.5.23 Is the following continuous at \((0,0)\)?

(a) \[ \frac{1}{x^2 + y^2 + 1} \] We know that \( \lim_{(x,y) \to (0,0)} \frac{1}{x^2 + y^2 + 1} = 1 \), because this is a rational function whose denominator does not vanish at \((0,0)\).

(b) \[ \sqrt{1 - x^2 - y^2} \] Since at \((0,0)\) \( 1 - x^2 - y^2 \) is positive, there is no trouble taking the square root. \( \lim_{(x,y) \to (0,0)} \sqrt{1 - x^2 - y^2} = 1 \).

(c) \[ \frac{\sqrt{x^2 + y^2}}{|x| + |y|^{1/3}} \] Look at approach along the \( x \)-axis: we get \( \lim_{x \to 0} \frac{|x|}{|x|} = 1 \). Approach along \( y \)-axis: get \( f = |y|^{2/3} \), which goes to zero. Not continuous at the origin, and there’s nothing to do.

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(d) \((x^2 + y^2) \ln(x^2 + 2y^2)\) is continuous at \((0, 0)\). Use L’Hopital’s rule! I forgot this trick in class, but it’s the nicest way to do things: recall that 

\[
\lim_{u \to 0} u \ln u = \lim_{u \to 0} \frac{\ln u}{1/u} = \lim_{u \to 0} \frac{1/u}{1/u^2} = \lim_{u \to 0} -u = 0.
\]

How can we apply this here? Use inequalities (’squeeze theorem’). When \(0 < x^2 + 2y^2 < 1\),

\[
0 > (x^2+y^2) \ln(x^2+2y^2) \geq (x^2+y^2) \ln(2(x^2+y^2)) = (x^2+y^2) \ln(x^2+y^2) + (x^2+y^2) \ln 2
\]

We know \((x^2 + y^2) \ln 2\) tends to zero as we approach the origin, and we can use the trick for the other term. As our desired function is ’squeezed’ between zero and something that approaches zero, it too must approach zero.

Or do it by hand.

\[
\lim_{(x,y) \to (0,0)} (x^2 + y^2) \ln(x^2 + 2y^2) = \lim_{(x,y) \to (0,0)} \frac{\ln(x^2 + 2y^2)}{1/(x^2 + y^2)}
\]

\[
= \lim_{(x,y) \to (0,0)} \frac{1/(x^2 + 2y^2)}{-1/(x^2 + y^2)^2} \cdot (2x + 4y)
\]

\[
= \lim_{(x,y) \to (0,0)} \frac{(x + 2y) \cdot (x^2 + y^2)^2}{(x + y) \cdot (x^2 + 2y^2)}
\]

\[
= 0
\]

where we get 0 by realizing that the numerator has a higher degree than the denominator in both \(x\) and \(y\).

(e) One would hope we could apply similar logic here. Let’s try:

\[
\lim_{(x,y) \to (0,0)} (x^2 + y^2) \ln |x + y| = \lim_{(x,y) \to (0,0)} \frac{\ln |x + y|}{1/(x^2 + y^2)}
\]

\[
= \lim_{(x,y) \to (0,0)} \frac{1/|x + y|}{-1/(x^2 + y^2)^2} \cdot (2x + 2y)
\]

\[
= \lim_{(x,y) \to (0,0)} -\frac{x^2 + y^2}{|x + y| \cdot (2x + 2y)}
\]

Now, the top and bottom have the same degree. Try approaching along \(x = y\):

\[
\lim_{x \to 0} -\frac{2x^2}{8|x|x} = \lim_{x \to 0} \pm \frac{1}{4}.
\]

Since approaching from \(x\) negative gives us a different value than approaching from \(x\) positive, this is not continuous at \((0, 0)\).

There are other ways to show that this is not continuous, as well.