1. How many elements in the symmetric group $\Sigma_6$ have order 6?

2. Let $D_{2n}$ be the dihedral group with $2n$ elements:

   $$D_{2n} = \langle a, b \mid a^n = e, b^2 = e, bab = a^{-1} \rangle$$

   Find necessary and sufficient conditions on an integer $k$ so that the two reflections $b$ and $ba^k$ generate the whole group: every element in $D_{2n}$ can be obtained by multiplying together copies of $b$ and $ba^k$ in some order.

3. Find all possible values of $x$ which are solutions to the following equations in modular arithmetic:

   (a) $x^2 = 1$ in $\mathbb{Z}/5$.
   (b) $x^2 = -1$ in $\mathbb{Z}/5$.
   (c) $x^2 + x + 1 = 0$ in $\mathbb{Z}/7$.
   (d) $x^3 + x^2 - 2x - 1 = 0$ in $\mathbb{Z}/13$.

4. For which prime numbers $p$ is the matrix

   $$\begin{bmatrix}
   1 & 1 & 2 \\
   1 & 2 & 3 \\
   2 & 3 & 47 \\
   \end{bmatrix}$$

   an element of $GL_3(\mathbb{Z}/p)$?

5. Let $F$ be a field and $V$ a finite-dimensional vector space over $F$. The dual space of $V$, called $V^*$, is the set of linear transformations $T : V \rightarrow F$. We define addition and scalar multiplication on $V^*$ as follows:

   (a) $(T_1 + T_2)(v) = T_1(v) + T_2(v)$
   (b) $(a \cdot T)(v) = a(T(v))$

   Show that these rules make $V^*$ into a vector space over $F$, of the same dimension as $V$. 