The exam itself will be closed book, no notes.

Note: There are more practice questions appearing here than would appear on an actual exam. The actual exam will have **five** questions, and **two** of them will be off this list.

Solutions will be posted on Monday, April 28.

1. Find all possible trajectories of the vector field \( w(x, y) = (-y, x) \) on \( \mathbb{R}^2 \).

2. If the first fundamental form in coordinates is given by \( E = e^u, F = 0, G = e^v \), find a vector field of unit length perpendicular to the vector field \( x_u - x_v \).

3. If \( f : S_1 \rightarrow S_2 \) is an isometry between surfaces and \( \alpha(s) : (a, b) \rightarrow S_1 \) is a geodesic parametrized by arc length, show that \( f(\alpha(s)) \) is also a geodesic parametrized by arc length.

4. Suppose \( x \) is a coordinate chart on a surface, with coefficients \( E, F, \) and \( G \) of the first fundamental form. Prove the following identities.

\[
\langle x_{uu}, x_u \rangle = \frac{1}{2} E_u \\
\langle x_{uu}, x_v \rangle = F_u - \frac{1}{2} E_v
\]

Use these to show the matrix identity

\[
\begin{bmatrix}
\frac{1}{2} E_u \\
F_u - \frac{1}{2} E_v
\end{bmatrix} =
\begin{bmatrix}
E & F \\
F & G
\end{bmatrix}
\begin{bmatrix}
\Gamma^1_{11} \\
\Gamma^2_{11}
\end{bmatrix}
\]

5. Prove that the sphere of radius \( R > 0 \) centered at the origin has constant Gaussian curvature \( 1/R^2 \) and mean curvature \( -1/R \).

6. Suppose \( (u(s), v(s)) \) is a curve in \( \mathbb{R}^2 \) and \( x \) is a coordinate chart so that \( x(u(s), v(s)) \) is a curve parametrized by arc length. Write down the conditions on \( u \) and \( v \) necessary for this curve to be a geodesic in the surface.
7. Let \( \alpha(s) = (f(s), g(s)) \) be a curve in \( \mathbb{R}^2 \) parametrized by arc length, and consider the coordinate chart on the associated surface of revolution given by

\[
x(u, v) = (f(u) \cos v, f(u) \sin v, g(u)).
\]

Prove that for any fixed angle \( \theta \), the meridian

\[
\alpha(s) = (f(s) \cos \theta, f(s) \sin \theta, g(s))
\]

is a geodesic parametrized by arc length.

8. Explain the sequence of steps (without calculating anything) taken to derive the Mainardi-Codazzi equations relating Christoffel symbols to \( e, f, \) and \( g \) from the formulas for \( x_{uu}, x_{uv}, \) and \( x_{vv} \).

9. Find the absolute value of the geodesic curvature of the curve \((\cos t \cos \theta, \sin t \cos \theta, \sin \theta)\) on \( S^2 \) for any fixed value of \( \theta \).

10. On a sphere of radius \( R > 0 \), suppose that we have a triangle with three geodesic sides, with interior angles \( \theta_1, \theta_2, \) and \( \theta_3 \). Find the area of the triangle.

11. Show that on a surface of nonpositive curvature, there are no simple closed geodesics that bound simple regions.

12. Calculate the geodesic curvature of the circle \( z = h \) on the cone \( x^2 + y^2 = z^2 \). Explain how the Gauss-Bonnet theorem relates these for different values of \( h \).

13. Calculate the index of the critical point \((0, 0)\) of the vector field

\[
w(x, y) = (x^2 - y^2, 2xy)
\]
on \( \mathbb{R}^2 \).