Math 8301, Manifolds and Topology
Homework 10
Due in-class on **Friday, Dec 5**

1. Suppose that $X \subset \mathbb{R}^N$ is a subspace. For any $n \geq 0$, the set $C_n^{lin}(X)$ of *linear chains in* X is the free abelian group on continuous maps $\Delta^n \to X$ such that the underlying map $\Delta^n \to \mathbb{R}^N$ is linear (in the sense of preserving lines). Show that $C_n^{lin}(X) \subset C_n^{sing}(X)$ is a subcomplex.

2. Show that, for any simplex $\Delta \subset \mathbb{R}^N$ which is the convex hull of some set of points in general position, the map $C_n^{lin}(\Delta) \to C_n^{sing}(\Delta)$ induces an isomorphism on homology.

3. By contrast, calculate the homology groups $H_n^{lin}(S^{N-1})$.

4. Suppose that X is a space, and $U = \{U_i\}_{i \in I}$ is an open cover: a collection of open subsets of X with $X = \cup U_i$. The *Čech complex* is the simplicial complex whose vertices are elements $i \in I$, and whose faces are the subsets $\{i_1, \cdots, i_n\}$ such that $\cap U_i \neq \emptyset$.

   For any values of $n > 0$ and $\epsilon > 0$, the circle S^1 has an open cover by the sets
   $$U_i = \{e^{2\pi i t} \,|\, t \in (\tfrac{i-1}{n} - \epsilon, \tfrac{i}{n} + \epsilon)\}$$
   for $1 \leq i \leq n$. Calculate the homology groups of the associated Čech complex (which depend on n and ϵ).