Math 8301, Manifolds and Topology
Homework 4
Due in-class on **Friday, Oct 3**

1. Suppose $f(z)$ is a monic polynomial $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ whose coefficients are complex numbers. Define
$$S = \{z \in \mathbb{C} \mid f'(z) = 0\} \text{ and } T = f(S).$$
(T is called the set of *singular values* of f.) Use the inverse function theorem, and the fundamental theorem of algebra, to show that the restricted map
$$f : \mathbb{C} \setminus f^{-1}(T) \to \mathbb{C} \setminus T$$
is a covering map.

2. Suppose $p : Y \to X$ is a covering map, and $y \in Y$. Use the unique disc lifting property to prove that the map $p_* : \pi_1(Y, y) \to \pi_1(X, p(y))$ is injective, and that its image is a subgroup.

3. Suppose Z is path-connected, $f : Z \to X$ is continuous, and $p : Y \to X$ is a covering map. If we are given $z \in Z$ and $y \in Y$ such that $f(z) = p(y)$, use unique path lifting to show that there exists *at most* one continuous map $\tilde{f} : Z \to Y$ such that $p\tilde{f} = f$ and $\tilde{f}(z) = y$, and that its value at any point $z' \in Z$ is already determined by f, z, and y.

4. Explain why, when we described a surface S by a string of letters indicating edges, the letters represent elements in $\Pi_1(S)$ and the string of letters represents a relation in $\Pi_1(S)$.

5. Suppose $p : Y \to X$ is a covering map. Here are two plausible-sounding but false statements:

   - If X is an n-manifold, then Y is an n-manifold.
   - If Y is an n-manifold, then X is an n-manifold.

   However, a manifold is assumed to be Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$. In each of these two statements, explain which one of these three properties succeeds or fails to be transported along the covering map. (In at least one case constructing a counterexample turns out to be hard, and so a rough explanation will suffice.)