1. Show that the fundamental group of the \( n \)-sphere \( S^n \) is trivial for \( n > 1 \) by directly showing that any loop \( \gamma \) is homotopic to the trivial loop.

2. Now give a proof of the same using the Seifert-Van Kampen theorem.

3. Suppose \( f(z) \) is a monic polynomial \( z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) whose coefficients are complex numbers. Recall \( S^1 = \{ w \in \mathbb{C} \mid |w| = 1 \} \). Show that there is a sufficiently large real number \( R > 0 \) such that
   
   (a) \( f(z) \neq 0 \) when \( |z| = R \), and
   
   (b) the resulting function \( S^1 \to \mathbb{C} \setminus \{ 0 \} \), given by \( w \mapsto f(Rw) \), is homotopic to the map \( w \mapsto (Rw)^n \).

4. Suppose you are given a simplicial complex with a finite set \( V \) of vertices and set \( F \) of faces. Let \( X \) be the space you get by realizing this simplicial complex. For definiteness, we’ll let \( V \) be the vector space with basis \( V \), and define

\[
X = \bigcup_{U \in F} \left\{ \sum_{v \in U} t_v \cdot v \mid t_v \geq 0, \sum t_v = 1 \right\} \subset V.
\]

Show that none of the faces of dimension 3 or greater have any effect on the fundamental group: you can put them in or take them of \( F \) without changing \( \pi_1 \). (This is also true if the set is infinite.)

5. A \textit{graph} is a simplicial complex with only vertices and edges, i.e. where no faces have dimension higher than one. A \textit{tree} is a graph, with at least one vertex, such that for any vertices \( p \neq q \), there exists a \textit{unique} sequence \( e_1, e_2, \cdots, e_n \) of edges such that

   (a) \( e_i \neq e_j \) for \( i \neq j \),

   (b) \( e_i \) and \( e_{i+1} \) always share a common vertex,

   (c) \( p \) is a vertex of \( e_1 \), and

   (d) \( q \) is a vertex of \( e_n \).
Show that any tree gives rise to a space with trivial fundamental group. (If you want, you can instead show the stronger statement that this space is contractible.)