1. Suppose $f : H \to G$ is a group homomorphism. Show that the amalgamated product $G \ast_H \{e\}$ is always isomorphic to the quotient $G/N$, where $N$ is the normal subgroup generated by the image of $f$.

2. Suppose $X$ is path-connected and $p, q$ are points in $X$. Construct a new space $X'$ by taking a disjoint union of $X$ and $[0, 1]$, then gluing 0 to $p$ and 1 to $q$. Show that $\pi_1(X', p)$ is the free product $\pi_1(X, p) \ast \mathbb{Z}$. (Hint: Seifert-van Kampen is hard to use directly here. Start by finding a loop $S^1 \to X'$ and show $X'$ is a deformation retract of one obtained by gluing in $S^1 \times [0, 1]$ along $S^1 \times \{0\}$.)

3. Suppose $X$ is path-connected and $p, q$ are points in $X$. We know that $\pi_1(X, p)$ and $\pi_1(X, q)$ are isomorphic, but that this isomorphism depends on a choice of path from $p$ to $q$. Show that there is a canonical isomorphism between $\pi_1(X, p)_{ab}$ and $\pi_1(X, q)_{ab}$ (in the sense that it does not depend on any choices).

4. Express the abelianization of the group

$$\langle a, b, c \mid abc^4 = a^4 c^2 = a^2 b^8 c^8 = e \rangle$$

as a product of cyclic abelian groups. (You do not need to give explicit generators.)

(If you are using a double-sided printer, note that this is not the last problem on the assignment.)
5. ("Sometimes homotopies don’t preserve basepoints") Suppose we have spaces $X$ and $Y$, together with two continuous maps $f, g : X \to Y$ and a basepoint $x \in X$. Suppose that there is a homotopy $H : X \times [0, 1] \to Y$ starting at $f$ and ending at $g$, but that $H$ does not necessarily preserve the basepoint. Show that if we define

$$\alpha(t) = H(x, t)$$

then there is an identity

$$g_*([\gamma]) = [\alpha^{-1}] \ast f_*([\gamma]) \ast [\alpha]$$

and so an identification of maps $\pi_1(X, x) \to \pi_1(Y, g(x))$.

Use this to show that if $i : X \to Y$ and $j : Y \to X$ are maps such that $ij$ and $ji$ are each homotopic to the identity, then $i_* : \pi_1(X, x) \to \pi_1(Y, i(x))$ is an isomorphism.