In the following, an \textit{abstract} \(m\)-dimensional smooth manifold is an \(n\)-dimensional topological manifold \(M\) (Hausdorff, second countable, and every point has a neighborhood homeomorphic to an open subset of \(\mathbb{R}^m\)) together with a \textit{coordinate atlas}: a set

\[ A = \{ \varphi_\alpha : U_\alpha \to V_\alpha \}_\alpha \]

of “charts”: homeomorphisms from open subsets \(U_\alpha\) of \(M\) to open subsets \(V_\alpha\) of \(\mathbb{R}^m\). In addition, we require that

\begin{itemize}
  \item the \(U_\alpha\) cover \(M\) in the sense that \(M = \bigcup U_\alpha\), and
  \item the charts are \textit{compatible} in the sense that the maps
        \[ \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \]
        are smooth for all choices of \(\alpha\) and \(\beta\).
\end{itemize}

1. If \(M\) is an abstract smooth manifold, a function \(M \to \mathbb{R}^k\) is \textit{smooth at} \(p\) if it is smooth in coordinates: there exists a coordinate chart \(\varphi_\alpha : U_\alpha \to V_\alpha\) in the atlas with \(p \in U_\alpha\) such that the function \(f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}^k\) is smooth at \(p\). Show that this is independent of the choice of coordinate chart.

2. Show that an abstract smooth manifold \(M\) has an exhaustion by compact sets: there exists a sequence of compact subspaces \(K_1 \subset K_2 \subset \cdots\) such that \(M = \bigcup K_i\). (Hint: Second countability is critical here.)

3. Suppose \(M\) is an abstract smooth manifold, \(K \subset U \subset M\) an inclusion of a compact subset into an open subset. Suppose we have a smooth function \(f : U \to \mathbb{R}^k\). Show that there exists an smooth function \(g : M \to \mathbb{R}^k\) such that \(f|_K = g|_K\) and \(g|_{M \setminus U} = 0\).

4. Show that an abstract smooth \(M\) has a smooth function \(T : M \to \mathbb{R}\) such that \(T^{-1}((-\infty,r])\) is compact for any \(r \in \mathbb{R}\). (Hint: Use an exhaustion by compact sets.)
5. Suppose $f(x)$ is a smooth function $\mathbb{R} \to \mathbb{R}$ such that

(a) $f(x) = 0$ for $x \neq (-1, 1)$,

(b) $f(0) = 1$,

(c) $f(x) = f(-x),$

(d) $f'(x) < 0$ for $x \in (0, 1/2)$.

Suppose that we are given, for each integer $n$, a real number $a_n$ such that $a_n \neq a_{n+1}$. Construct a smooth function $g(x) : \mathbb{R} \to \mathbb{R}$ whose set of singular points is $\mathbb{Z}$ and such that $f(n) = a_n$. 