How to get character tables of symmetric groups

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These are some notes on how to construct character tables of symmetric groups. The focus isn't going to be on *why* it works, but rather *how*.

Recall that the conjugacy classes of the symmetric group S_n were in correspondence with partitions

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r,$$

which satisfy

$$\sum \lambda_i = n.$$

First, we can put a *partial order* on partitions. Say that we have two partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ and $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$. We say that $\lambda \geq \mu$ if

$$\begin{aligned} \lambda_1 &\geq \mu_1 \\ \lambda_1 + \lambda_2 &\geq \mu_1 + \mu_2 \\ \lambda_1 + \lambda_2 + \lambda_3 &\geq \mu_1 + \mu_2 + \mu_3 \\ &\vdots \end{aligned}$$

In terms of Young diagrams, this means that you can get the Young diagram for μ by taking the Young diagram for λ and moving blocks downwards.

Here are the partitions of 6, listed in decreasing order from left to right.

$$\begin{tikzcd}
 & & & 3,3 \arrow[dr] & & 2,2,2 \arrow[dr] & & & \\
6 \arrow[r] & 5,1 \arrow[r] & 4,2 \arrow[ur] \arrow[dr] & & 3,2,1 \arrow[ur] \arrow[dr] & & 2,2,1,1 \arrow[r] & 2,1,1,1,1 \arrow[r] & 1,1,1,1,1,1 \\
 & & & 4,1,1 \arrow[ur] & & 3,1,1,1 \arrow[ur] & & &
\end{tikzcd}$$

Associated to a partition $\lambda = (\lambda_1, \dots, \lambda_r)$, we can define a permutation representation of S_n as follows.

Let $[n]$ be the set of integers $\{1, 2, \dots, n\}$, and let X_λ be the set of all possible ways to divide $[n]$ into sets of size λ_1, λ_2, λ_3, …, λ_r.

For example, $X_{5,1}$ is the set of all ways to divide [6] into a set of size 5 and a set of size 1. There are 6 of these:

$$X_{5,1} = \Big\{(\{1,2,3,4,5\},\{6\}),(\{1,2,3,4,6\},\{5\}),(\{1,2,3,5,6\},\{4\}),\\ (\{1,2,4,5,6\},\{3\}),(\{1,3,4,5,6\},\{2\}),(\{2,3,4,5,6\},\{1\})\Big\}.$$

Note that all we really need to do is pick a single element (the set of size 1), and then the set of size 5 is everything that's left over.

Sometimes we prefer to write these as *tabloids*. A tabloid of shape λ (or λ-tabloid) is a way to put all the numbers $1, \dots, n$ into the Young diagram for λ, except that order doesn't matter in the rows. Here are some example $(3,3)$-tabloids.

$$\begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} = \begin{array}{ccc} \hline 3 & 2 & 1 \\ \hline 4 & 6 & 5 \\ \hline \end{array} \neq \begin{array}{ccc} \hline 4 & 2 & 1 \\ \hline 3 & 6 & 5 \\ \hline \end{array}$$

The symmetric group acts on X_λ, the set of all paritions of $[n]$ of shape λ, by acting on the elements. You can think of it as acting on the tabloids of shape λ by permuting the entries. This permutation representation gives rise to a character σ_λ.

Let's write the characters of these permutation representations in a table for S_4. To do this, we have to do some work and figure out how many tabloids are fixed by each cycle type; here is the answer.

	$1,1,1,1$	$2,1,1$	$2,2$	$3,1$	4
σ_4	1	1	1	1	1
$\sigma_{3,1}$	4	2	0	1	0
$\sigma_{2,2}$	6	2	2	0	0
$\sigma_{2,1,1}$	12	2	0	0	0
$\sigma_{1,1,1,1}$	24	0	0	0	0

The top row lists the different conjugacy classes by way of listing partitions; the left-hand edge lists the different representations we get for different shapes of tabloid. The top line is the trivial representation, while the bottom one is the regular represenation.

Note that we wrote the conjugacy classes so that they are in increasing order from left to right, and the partitions so that they decrease from top to bottom. This is important, or the method doesn't work! Also, notice all the zeros in the bottom right.

Now we're going to do something like row reduction.

- The trivial representation is already irreducible, so the top row is an irreducible character; let's call it $\chi_4 = \sigma_4$.

- We can figure out how many copies of χ_4 each of the lower characters contains by taking inner products.

$$\begin{aligned}\langle \chi_4, \sigma_{3,1} \rangle &= 1\\ \langle \chi_4, \sigma_{2,2} \rangle &= 1\\ \langle \chi_4, \sigma_{2,1,1} \rangle &= 1\\ \langle \chi_4, \sigma_{1,1,1,1} \rangle &= 1\end{aligned}$$

- Then, since we know how many copies of χ_1 occur in the lower representations, we can subtract them off and get a new table:

	$1,1,1,1$	$2,1,1$	$2,2$	$3,1$	4
χ_4	1	1	1	1	1
$\sigma'_{3,1}$	3	1	-1	0	-1
$\sigma'_{2,2}$	5	1	1	-1	-1
$\sigma'_{2,1,1}$	11	1	-1	-1	-1
$\sigma'_{1,1,1,1}$	23	-1	-1	-1	-1

- Here's the miracle: Now row 2 is an irreducible character $\chi_{3,1}$; you can see this by taking its inner product with itself.
- We can now repeat by taking the inner product of $\chi_{3,1}$ with the σ characters and subtracting them off.

$$\begin{aligned}\langle \chi_{3,1}, \sigma_{2,2} \rangle &= 1\\ \langle \chi_{3,1}, \sigma_{2,1,1} \rangle &= 2\\ \langle \chi_{3,1}, \sigma_{1,1,1,1} \rangle &= 3\end{aligned}$$

	$1,1,1,1$	$2,1,1$	$2,2$	$3,1$	4
χ_4	1	1	1	1	1
$\chi_{3,1}$	3	1	-1	0	-1
$\sigma''_{2,2}$	2	0	2	-1	0
$\sigma''_{2,1,1}$	5	-1	1	-1	1
$\sigma''_{1,1,1,1}$	14	-4	2	-1	2

- Once again, something mysterious has happened, and row 3 is irreducible. Let's call it $\chi_{2,2}$, and subtract it off from the lower rows.

$$\begin{aligned}\langle \chi_{2,2}, \sigma_{2,1,1} \rangle &= 1\\ \langle \chi_{2,2}, \sigma_{1,1,1,1} \rangle &= 2\end{aligned}$$

	$1,1,1,1$	$2,1,1$	$2,2$	$3,1$	4
χ_4	1	1	1	1	1
$\chi_{3,1}$	3	1	-1	0	-1
$\chi_{2,2}$	2	0	2	-1	0
$\sigma'''_{2,1,1}$	3	-1	-1	0	1
$\sigma'''_{1,1,1,1}$	10	-4	-2	1	2

- As you might guess by now, the new row 4 is irreducible, so we can call it $\chi_{2,1,1}$ and subtract it off from the last row.

$$\langle \chi_{2,1,1}, \sigma_{1,1,1,1} \rangle = 3$$

	$1,1,1,1$	$2,1,1$	$2,2$	$3,1$	4
χ_4	1	1	1	1	1
$\chi_{3,1}$	3	1	-1	0	-1
$\chi_{2,2}$	2	0	2	-1	0
$\chi_{2,1,1}$	3	-1	-1	0	1
$\chi_{1,1,1,1}$	1	-1	1	1	-1

And surprisingly enough, we've ended with the character table of S_4. This method works to construct the character table of S_n for any n. *Why* this method works is a bit of a mystery that we'll have to reserve for later.

You might notice that we did a lot of work to figure out the bottom row of the table, even though it just turned out to be the sign representation. There's also some other almost-symmetry to the table. But we'll save that for another day.