1. A simplicial complex consists of a pair \((V, F)\), where \(V\) is a set of vertices and \(F\) (the set of faces) is a collection of finite subsets of \(V\) satisfying the following properties.

- We have \(\{v\} \in F\) for all \(v \in V\).
- If \(S \subset T\) and \(T \in F\), then \(S \in F\).

(Course 6 people might know this as a “hereditary hypergraph”.) From a simplicial complex \((V, F)\), we form a space \(X\) by starting with the vertices \(V\) and, for every \(S \in F\) of size \(n+1\), we glue in a unique \(n\)-simplex whose vertices are the elements of \(S\). The faces of this simplex correspond to the subsets of \(S\).

More precisely, let \(W\) be the vector space with basis \(\{e_v | v \in V\}\), and let

\[
X = \bigcup_{S \in F} \left\{ \sum_{v \in S} t_v e_v \middle| 0 \leq t_v \leq 1, \sum t_v = 1 \right\}.
\]

Suppose that we have chosen a partial order on \(V\) such that for any \(S \in F\), the elements of \(S\) are totally ordered. Use this to give a \(\Delta\)-complex structure on \(X\). (You may assume that for any \(v_0, \ldots, v_n\) in a vector space \(W\), there is a unique affine transformation \(f: \Delta^n \to W\) such that \(f\) takes the \(i\)th vertex of \(\Delta^n\) to \(v_i\).)

**Update.** It has been pointed out to me that I need to be explicit about what the topology on the vector space \(W\) is; it’s not the metric topology or the product topology. The topology on \(W\) is a limit topology: A subspace \(A \subset W\) is closed if and only if \(A \cap U\) is closed for any finite-dimensional subspace \(U\) of \(W\).


3. In class, we defined face maps \(d^i_n : \Delta^{n-1} \to \Delta^n\) for \(0 \leq i \leq n\), and subdivision maps \(s^j_n : \Delta^{n+1} \to \Delta^n \times [0, 1]\) for \(1 \leq j \leq n+1\). These satisfy the following relations.

\[
\begin{align*}
    s^j_n \circ d^i_{n+1} &= \begin{cases} (d^{i-1}_n, id) \circ s^j_{n-1} & \text{if } j < i \\ (d^i_n, id) \circ s^j_{n-1} & \text{if } j > i + 1 \end{cases} \\
    s^1_n \circ d^0_{n+1} &= (id, 1) \\
    s^{n+1}_n \circ d^{n+1}_{n+1} &= (id, 0) \\
    s^i_n \circ d^i_{n+1} &= s^{i+1}_n \circ d^i_{n+1}
\end{align*}
\]
If $H : X \times [0,1] \to Y$ is a homotopy between the maps $f$ and $g$, we then defined a homotopy operator $h : C_n(X) \to C_{n+1}(Y)$ by

$$h(\sum n_\sigma[\sigma]) = \sum n_\sigma \sum_{j=1}^{n+1} (-1)^j [H \circ (\sigma, \text{id}) \circ s_n^j]$$

where the map $[H \circ (\sigma, \text{id}) \circ s_n^j]$ is the composite map $\Delta^{n+1} \to Y$.

Use the given relations to show that for any $\sigma : \Delta^n \to X$, we have

$$\partial h(\sigma) = f_*(\sigma) - g_*(\sigma) - h(\partial \sigma)$$

in $C_{n+1}(Y)$.

4. Suppose $f : A \to B$ and $g : B \to C$ are homomorphisms of abelian groups. Show that there is an exact sequence

$$0 \to \ker(f) \to \ker(gf) \to \ker(g) \to \coker(f) \to \coker(gf) \to \coker(g) \to 0.$$