1. Quickies.

- Show that the Hurewicz map \( \pi_n(X, x) \to \tilde{H}_n(X) \cong H_n(X, x) \) is a group homomorphism for \( n > 0 \). (Hint: Consider the effect on homology of the pinch map \( S^n \to S^n \vee S^n \).

- Prove the lemma mentioned in class: If \( A \) is a set with two binary operations \( \ast, \circ \) that have the same identity element \( e \in A \) and satisfy an interchange law
  \[(a \ast b) \circ (c \ast d) = (a \circ c) \ast (b \circ d),\]
  show that \( \ast = \circ \) and that \( a \ast b = b \ast a \) for all \( a, b \in A \).

2. Suppose that a group \( G \) acts properly discontinuously on a space \( Y \) on the left, and suppose that \( A, B \) are discrete sets with right actions of \( G \).

We can construct a space
\[ A \times_G Y = A \times Y/\{(ag, y) \sim (a, gy) \mid g \in G\}. \]
and similarly for \( B \). Suppose \( f : A \to B \) satisfies \( f(ag) = f(a)g \) for all \( a \in A, g \in G \). Show that there is an induced covering map \( A \times_G Y \to B \times_G Y \).

Give an explicit description of the preimage of a point of \( B \times_G Y \).

3. A fiber bundle with fiber \( F \) is a map \( p : E \to B \) of spaces such that, for every \( b \in B \), there exists a neighborhood \( U \) of \( b \) and a homeomorphism \( \phi : U \times F \to p^{-1}U \) such that \( p\phi(u, f) = u \) for all \( (u, f) \in U \times F \).

Show that fiber bundles have a disc lifting property, as follows. Let \( D^{n-1} \subset D^n = [0, 1]^n \) be the “cap”, consisting of the union of all faces but one. Suppose that we have a map \( g : D^n \to B \) which has a chosen lift on the cap \( \tilde{g} : D^{n-1} \to E \). Show that there exists an extension (not necessarily unique) to a lift \( \tilde{g} : D^n \to E \). (Hint: Subdivide, and use induction on \( n \).)

4. In this exercise, we will show some “exactness” properties of the long exact sequence in homotopy in low degrees. Recall that \( \pi_1(X, A, x) \) is acted on by the group \( \pi_1(X, x) \) by path composition: if \( \gamma \) is a loop based at \( x \) and \( \lambda \) is a path starting at \( x \) and ending in \( A \), we can form the composition \( \gamma \lambda \).

In particular, the image of \( \gamma \) in \( \pi_1(X, A, x) \) is \( \gamma \) times the trivial element.

Show that two elements \( \gamma, \gamma' \) of \( \pi_1(X, x) \) have the same image in \( \pi_1(X, A, x) \) if and only if \( \gamma = \gamma' \alpha \) for some \( \alpha \) in the image of \( \pi_1(A, x) \).

Show that two elements \( \lambda, \lambda' \) in \( \pi_1(X, A, x) \) have the same image in \( \pi_0(A) \) if and only if \( \lambda = \gamma \lambda' \) for some \( \gamma \in \pi_1(X, x) \).

Show that an element of \( \pi_0(A) \) maps to the component of \( x \) in \( \pi_0(X) \) if and only if it lifts to \( \pi_1(X, A, x) \).