18.906 Problem Set 3
Due Wednesday, February 28 in class

1. Quickies.
   - Let $X = \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$. Show that the inclusion $\{0\} \hookrightarrow X$ is not a cofibration.
   - Show that if $A \to B$ is a cofibration of compactly generated Hausdorff spaces, so is $A \times [0,1] \to B \times [0,1]$.

2. Suppose $X$ is a space and $f, g : S^n \to X$ are freely homotopic maps (meaning that we do not require the homotopy to preserve the basepoint). Let $Cf$ and $Cg$ be the mapping cones of $f$ and $g$, formed by attaching $(n+1)$-cells using the attaching maps $f$ and $g$.

   Explicitly show that we can construct maps $\phi : Cf \to Cg$ and $\psi : Cg \to Cf$ such that $\phi|_X = \psi|_X = id_X$, together with homotopies $H$ from $\phi \circ \psi$ to $id_{Cg}$ and $H'$ from $\psi \circ \phi$ to $id_{Cf}$ which restrict to the constant homotopy on $X$.

3. Suppose that $X$ is a space with basepoint $x$, and we have two based maps $f, g : S^n \to X$. Show that $f$ and $g$ are freely homotopic if and only if there exists an element $\gamma \in \pi_1(X,x)$ such that the action of $\gamma$ on $[f] \in \pi_n(X,x)$ gives $[g] \in \pi_n(X,x)$.

4. In this exercise we will construct a family of fibrations that don’t look much like fiber bundles.

   Suppose that $B = U \cup V$ for open subsets $U$ and $V$. Define a space

   $$E = \{(b,t) \in B \times [0,1] \mid t = 0 \text{ if } b \notin U, \ t = 1 \text{ if } b \notin V\}.$$ 

   $E$ is formed by gluing $[0,1]$ times $U \cap V$ to $U$ and $V$ at the ends.

   Show that the obvious projection map $p : E \to B$ is an acyclic Serre fibration: if we have a map $f : D^n \to B$ and a map $\tilde{f} : \partial D^n \to E$ such that $p\tilde{f} = f|\partial D^n$, show that there exists an extension $\tilde{f} : D^n \to E$ such that $p\tilde{f} = f$. (Hint: You might need a theorem from elementary point-set topology.)

   Show that an acyclic Serre fibration $p$ automatically induces isomorphisms $p_* : \pi_n(E,e) \to \pi_n(B,p(e))$ for all basepoints $e \in E$, $n > 0$, and an isomorphism $\pi_0(E) \to \pi_0(B)$. 