18.906 Problem Set 5

Due Wednesday, March 14 in class

Questions 1-3 are about the following theorem, usually known as the Brown Representability Theorem.

Theorem. Suppose F is a contravariant functor from the *homotopy category* of based spaces to the category of based sets (in particular, F takes homotopy equivalences to isomorphisms) satisfying the following properties.

- (Wedge axiom) For any based spaces, the natural map $F(\bigvee X_\alpha) \to \prod F(X_\alpha)$ induced by the inclusions $X_\alpha \to \bigvee X_\alpha$ is an isomorphism.
- (Mayer-Vietoris axiom) If X is a based CW-complex which is the union of subcomplexes U and V, the sequence of sets

  $$F(X) \to F(U) \times F(V) \to F(U \cap V)$$

  is exact, in the sense that if $s_1 \in F(U)$ and $s_2 \in F(V)$ have the same image in $F(U \cap V)$, there exists an element $s \in F_(X)$ such that the image of s is (s_1, s_2).

Then there exists a based space Y and an element $\eta \in F(Y)$ such that for all finite CW-complexes X, the map

$$\begin{aligned} \eta_X : [X, Y]_* &\to F(X) \\ f &\mapsto F(f)(\eta) \end{aligned}$$

is an isomorphism of sets, so F is isomorphic to a representable functor.

For problems 1-3, assume that we have a functor F satisfying the wedge and Mayer-Vietoris axioms. (Note that $[-, Y]_*$ automatically satisfies these axioms for any space Y.) We will show the inductive portion of the proof.

1. Use the wedge axiom to show $F(*)$ is a one-point set, and use this to define a canonical "trivial" element in $F(X)$ for all X. Then use the Mayer-Vietoris axiom to show that if $A \subset X$ is a subcomplex, the sequence of maps

   $$F(X/A) \to F(X) \to F(A)$$

   is exact, i.e. that an element of $F(X)$ maps to the trivial element in $F(A)$ if and only if it lifts to $F(X/A)$. (Hint: Mapping cones.)

2. Show that one can construct a space Y with an element $\eta in F(Y)$ such that the map $\eta_X : [X, Y]_* \to F(X)$ is an epimorphism for all CW complexes X with dimension less than d.

3. Suppose Y is a space and $\eta \in F(Y)$ such that the map $\eta_X : [X, Y]_* \to F(X)$ is an epimorphism for all finite CW-complexes X and an isomorphism for all CW-complexes of dimension less than d. Show that one can form a new space Y' by attaching $(d+1)$-cells to Y, with an element $\eta' \in F(Y')$, such that the map η'_X is an isomorphism for all CW-complexes k of dimension less than or equal to d. (Hint: Consider pairs of maps $f, g : X \to Y$ with $\eta_X(f) = \eta_X(g)$; construct a homotopy from f to g on the $(d-1)$-skeleton and attach cells to allow an extension to the d-skeleton.)

4. Suppose $f : X \to Y$ is a map of 1-connected CW-complexes such that the induced map $f_* : H_*(X) \to H_*(Y)$ is an isomorphism. Show that f is a homotopy equivalence. (Hint: Replace f with a cofibration and identify the first nonvanishing relative homotopy group.)