Truncated Brown-Peterson spectra

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Conjecture (Ausoni-Rognes)

For any prime $p$ and $n \geq 0$, there exist localization sequences

\[ K(BP\langle n - 1 \rangle_p^\wedge)_p \rightarrow K(BP\langle n \rangle_p^\wedge)_p \rightarrow K(E(n)_p^\wedge)_p. \]

Here $BP\langle k \rangle$ and $E(n)$ are $p$-local truncated Brown-Peterson spectra and Johnson-Wilson spectra respectively.

- $n = 0$ is a devissage result of Quillen
- $n = 1$ is a theorem of Blumberg-Mandell
Things to understand

- More multiplicative structure on $R$ gives more structure to $K(R)$ and to a zoo of related objects: $TC$, $TR$, $TF$, $THH$
- This makes computations in algebraic $K$-theory easier by imposing multiplication and power operations
- Can we understand these truncated Brown-Peterson spectra $BP\langle n \rangle$?
- Can they be equipped with extra multiplicative structure?
A formal group law $\mathbb{G}$ over a ring $R$ is a power series $x + \mathbb{G} y$ in $R[[x, y]]$ satisfying power series identities:

- **unitality:** $x + \mathbb{G} 0 \equiv x$
- **commutativity:** $x + \mathbb{G} y \equiv y + \mathbb{G} x$
- **associativity:** $(x + \mathbb{G} y) + \mathbb{G} z \equiv x + \mathbb{G} (y + \mathbb{G} z)$

- The existence of an “inverse” is automatic
- Underlying any formal group law is a *formal group*, which remembers only the isomorphism type
Suppose $R$ is a torsion-free ring. A formal group law $\mathbb{G}$ over $R$ is $p$-typical if there is a power series

$$\ell(x) = x + \ell_1 x^p + \ell_2 x^{p^2} + \cdots,$$

with coefficients in $R \otimes \mathbb{Q}$, such that

$$x +_{\mathbb{G}} y \equiv \ell^{-1}(\ell(x) + \ell(y)).$$

Such a power series is called a \textit{logarithm} for $\mathbb{G}$.

- This has an intrinsic definition, applicable over any ring
- Every formal group law over a $p$-local ring is isomorphic to a $p$-typical one
**Definition**

A *complex oriented cohomology theory* is

- a cohomology theory $E^*$,
- with an associative and commutative multiplication,
- and an element $x \in \tilde{E}^2(\mathbb{CP}^\infty)$ which restricts to the element $1 \in \tilde{E}^2(S^2)$.

- Since $\mathbb{CP}^\infty$ classifies complex line bundles, $E^*$ gets a natural characteristic class $c_1(L)$ for complex line bundles
- This automatically extends to a full theory of Chern classes
Chern classes produce formal group laws

**Proposition**

Given a complex oriented cohomology theory, there is a formal group law $G_E$ over $E^*$ such that the first Chern class satisfies a natural identity

$$c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + G_E c_1(\mathcal{L}') .$$

- Different choices of orientation produce different, but isomorphic, formal group laws
- If we are feeling energetic, we can throw gradings into the story
Example: Complex cobordism

The cohomology theory $MU^*$, associated to bordism of stably almost-complex manifolds, is complex orientable.

**Theorem (Milnor)**

$MU^* \cong \mathbb{Z}[b_1, b_2, \ldots]$, where $b_i$ is in grading $2i$.

**Theorem (Quillen)**

*The formal group law $G_{MU}$ is universal.*

This means that, for any ring $R$, there is a natural bijection

$$\{\text{homomorphisms } MU^* \to R\} \quad \longleftrightarrow \quad \{\text{formal group laws over } R\}$$
Brown-Peterson cohomology

- Brown-Peterson cohomology $BP^*$ is a $p$-local, complex orientable cohomology theory.
- The coefficient ring is isomorphic to $\mathbb{Z}(p)[v_1, v_2, \ldots]$ with $v_i$ in degree $(2p^i - 2)$.
- $G_{BP}$ is $p$-typical.
- $G_{BP}$ is universal among $p$-local, $p$-typical formal group laws.
- There is a map $MU^* \rightarrow BP^*$ classifying the inclusion

$$\{p\text{-typical formal group laws}\} \subset \{\text{formal group laws}\}$$
Proto-definition

\[ BP\langle n \rangle^* \text{ is a complex oriented cohomology theory whose underlying coefficient ring is the quotient } \mathbb{Z}_\langle p \rangle[v_1, v_2, \ldots, v_n] \text{ of } BP^*. \]

- Classically constructed using Baas-Sullivan theory of manifolds with singularity
- Newer constructions, with more structure, using more machinery
- The \( v_i \) are not intrinsically defined and so the definition depends (at least) on a choice of sequence of generators (e.g. Hazewinkel vs. Araki)
Generalized $BP\langle n \rangle$

A cohomology theory $R^*$ is represented by a spectrum $R$.

**Proposition**

The following are equivalent for a spectrum $R$ representing a cohomology theory $R^*$ with a commutative and associative multiplication.

1. $R$ admits a $p$-typical orientation so that the map $BP^* \to R^*$ maps the (intrinsic) subring $\mathbb{Z}(p)[v_1, \ldots, v_n] \subset BP^*$ isomorphically to $R^*$.

2. $R$ is a $p$-local, connective, finite type spectrum such that $H^*(R; \mathbb{F}_p)$ is isomorphic to the quotient $\mathcal{A}^*/(Q^0, Q^1, \ldots, Q^n)$ of the Steenrod algebra.

We will call such a spectrum a generalized truncated Brown-Peterson spectrum.
Questions about generalized $BP\langle n \rangle$

Things we don’t appear to know about such spectra:

- For a given quotient $\mathbb{Z}_<(p)[v_1, \ldots, v_n]$ of $MU^*$, how many distinct complex oriented cohomology theories are there with this given formal group law?
- Given distinct such formal group laws, when do the spectra have the same underlying homotopy type?
- Does any $(p$-local, finite type) spectrum with this homology automatically have a ring structure?

Standard technology (the Adams spectral sequence) appears to be very messy as soon as $n > 1$. 
Power operations

- Multiplication on the cohomology theory might be lifted to a strictly commutative multiplication on the spectrum level.
- This provides extra power operation structure.
- Given $\alpha: X \to R$, we get a factorization.

\[
\begin{array}{c}
X \\ \downarrow \\
X \times B\Sigma_p \\
\Delta \\
X^p \\
\downarrow \\
X^p / \Sigma_p \\
\longrightarrow \\
R
\end{array}
\]
Subgroups produce power operations

- Study of these power operations and formal group data initiated in Ando’s thesis.
- If \( R \) is strictly commutative and complex orientable, the power operations equip \( \mathbb{G}_R \) with *quotient operations*.
- Given a ring map \( f: R^* \to S \) and a subgroup \( H \subset f^*(\mathbb{G}_R) \) of the formal group, we get a new ring homomorphism \( f_H: R^* \to S \) and a map \( f^*(\mathbb{G}_R) \to (f_H)^*(\mathbb{G}_R) \) with kernel \( H \).
- In practice, if \( R^* \) parametrizes some type of object \( X \) with an attached formal group, then this means that there is a canonical way to take quotients of the formal group while producing new objects of type \( X \).
Situations of interest

- \( MU \) has a strictly commutative structure, and \( MU^* \) parametrizes formal group laws; Ando calculated how the power operations give a canonical formal group law on any quotient formal group.
- \( BP^* \) parametrizes \( p \)-typical formal group laws; a quotient is isomorphic to a \( p \)-typical law, but there is no reason to expect compatibility with the canonical law on a quotient (\( MU \) and \( p \)-typical \( BP \) are incompatible — work of Noel-Johnson).
- Generalized \( BP\langle n \rangle \) parametrizes formal group laws of a restricted “shape”; there is no intrinsic description, so we don’t even know if these types of formal groups are closed under quotients!
Realizability for $BP\langle 2 \rangle$

There are some positive results for $BP\langle 2 \rangle$.

**Theorem (L.-Naumann)**

Let $G$ be a formal group law over the ring $R^* = \mathbb{Z}_p[v_1, v_2]$ which might come from a generalized $BP\langle 2 \rangle$. Then there is a strictly commutative ring spectrum $R$ realizing this formal group law data if and only if the subring

$$\mathbb{Z}[v_1^{p+1}/v_2]_p \subset \mathbb{Z}((v_2/v_1^{p+1}))_p$$

is closed under a certain algebraic power operation $\theta$ on the right-hand side.

Any two such commutative objects are equivalent.

The proof is mainly $K(1)$-local obstruction theory along the lines of the “old” construction of $tmf$. 
Elliptic curves with level 3 structure

Theorem (L.-Naumann)
There exists a strictly commutative generalized $BP\langle 2 \rangle$ at the prime 2.

- To get this, consider the elliptic curve
  \[ y^2 + v_1 xy + v_2 y = x^3 \]
  over $\mathbb{Z}_2[v_1, v_2]$, which parametrizes elliptic curves with a choice of 3-torsion point (after Rezk, Mahowald-Rezk)
- This elliptic curve produces a formal group law
- The universal property of this moduli object forces the existence of the power operation data
Realizing diagrams over the Steenrod algebra

**Theorem (L.-Naumann)**

There exists a commutative diagram of strictly commutative ring spectra realizing a classical diagram of modules over the Steenrod algebra:

\[
\begin{array}{ccc}
\text{tmf}_2 & \rightarrow & \text{ko}_2 \\
\downarrow & & \downarrow \\
\text{BP}_2 & \rightarrow & \text{ku}_2
\end{array}
\quad \begin{array}{ccc}
\mathbb{A}^*/\mathbb{A}(2) & \leftarrow & \mathbb{A}^*/\mathbb{A}(1) \\
\uparrow & & \uparrow \\
\mathbb{A}^*/\mathbb{E}(2) & \leftarrow & \mathbb{A}^*/\mathbb{E}(1)
\end{array}
\]

- The maps come from the interpretation in terms of moduli of elliptic curves
- Horizontal maps come from evaluating at a “ramified” cusp
Realization with a form of $K$-theory

- While we’re at it, there’s also an unramified cusp and we could use that instead to construct a similar diagram

\[
\begin{array}{ccc}
tmf_{(2)} & \rightarrow & ko_{(2)} \\
\downarrow & & \downarrow \\
BP \langle 2 \rangle & \rightarrow & ku^T_{(2)}
\end{array}
\]

- Here $ku^T$ is the form of $K$-theory associated with the formal group law

\[
x +_G y = (x + y + 3xy)/(1 - 3xy)
\]

- This becomes isomorphic to the multiplicative formal group after adjoining a third root of unity