Secondary power operations and the Brown–Peterson spectrum at the prime 2

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Abstract

The dual Steenrod algebra has a canonical subalgebra isomorphic to the homology of the Brown–Peterson spectrum. We will construct a secondary operation in mod-2 homology and show that this canonical subalgebra is not closed under it. This allows us to conclude that the 2-primary Brown–Peterson spectrum does not admit the structure of an $E_n$-algebra for any $n \geq 12$, answering a question of May in the negative.

1 Introduction

The following appeared as Problem 1 in J.P. May’s “Problems in infinite loop space theory” [May75].

Problem 1.0.1. For any prime $p$, does the $p$-local Brown–Peterson spectrum $BP$ of [BP66] admit the structure of an $E_\infty$-algebra?

Our goal in this paper is to address this question. We will construct a secondary operation in the homology of $E_\infty$-algebras at the prime 2 and show, with an analysis that begins with the calculations of Johnson–Noel [JN10], that the homology $H_\ast BP$ cannot admit such a secondary operation. Thus, Problem 1.0.1 has a negative answer at the prime 2.

1.1 Background

Coherently commutative multiplication structures have a long history in homotopy theory, originating in the study of the cup product. The cup product in the cohomology of a space $X$ comes from the structure of a differential graded algebra on the cochains $C^\ast(X)$, and while there are many variants on this algebra structure that all give rise to the cup product there is no natural cochain-level cup product that is graded-commutative. Instead, the cup product $\alpha \smile \beta$ and its reverse $\pm \beta \smile \alpha$ are chain homotopic by a natural operation $\alpha \smile_1 \beta$, called the cup-1 product. The cup-1 product

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is not graded-commutative either, but differs from its reverse by an operation called the
cup-2 product, and these constructions extend out both to arbitrarily high “coherences”
giving cup-i products for all i and to operations accepting arbitrarily many inputs
(giving a more complicated set of operations of several variables discussed in [MS04]).
The result is called an $E_\infty$-algebra structure and Steenrod’s reduced power operations
in the cohomology of spaces are built from it [Ste62].

Since then, these coherently commutative multiplications have been recognized in
many other areas: iterated loop spaces, monoidal structures on categories, structures
in mathematical physics related to string theory, and multiplications in cohomology
theories. By contrast with algebra, where commutativity is simply a property of a
ring, coherent multiplications come in a hierarchy: there are $E_1$-algebra structures that
correspond to associative products and there are $E_\infty$-algebra structures that correspond
to commutative products, but there are also $E_n$-algebra structures for $1 < n < \infty$ that
interpolate between these concepts.

When we switch from ordinary cohomology to generalized cohomology theories,
chain complexes become replaced with spectra. A ring spectrum $R$ is a representing
object for a cohomology theory $R^*(-)$ so that cohomology with coefficients in $R$
aturally takes values in rings. This was refined to the concept of an $E_n$-algebra
structure on the spectrum $R$ in [BMMS86, I.4], and these more refined algebras have
come to occupy a central role because $E_n$-algebra structures produce concrete tools
that are not available to an ordinary ring spectrum [Man12, Lur14].

- An $E_1$-algebra $R$ can be given categories of left $R$-modules and right $R$-modules,
  whose homotopy categories are triangulated categories. These enjoy several
  forms of compatibility as $R$ varies, extend to categories of bimodules, and have
  relative smash products $\wedge_R$ with properties much like the tensor product.

- The category of left modules over an $E_2$-algebra $R$ is canonically equivalent to the
  category of right modules, and the smash product $\wedge_R$ makes the category of left $R$
  modules into a monoidal category. The homotopy category of left $R$-modules has
  the structure of a (neither symmetric nor braided) tensor triangulated category.

- The homotopy category of left modules over an $E_3$-algebra has the structure of
  a braided monoidal category.

- The homotopy category of left modules over an $E_4$-algebra has the structure of
  a symmetric monoidal category.

- The category of modules over an $E_\infty$-algebra $R$ has homotopy-theoretic versions
  of symmetric power operations, making it possible to discuss a relative version of
  the above: we can define $E_n$-$R$-algebras which satisfy all of the above properties.

- An $E_\infty$-algebra $R$ has, for any principal $\Sigma_n$-bundle $P \to B$, natural geometric power
  operations $R^0(X) \to R^0(P \times_{\Sigma_n} X^n)$ and $R^0(X) \to R^0(B\times X)$ in $R$-cohomology that
  enhance the multiplicative structure. When $P = \Sigma_n$ these recover the operation
  that sends a class to its $n$’th power.

Many examples of $E_\infty$-algebras exist. Commutative rings $A$ produce $E_\infty$-algebras $HA$
via the Eilenberg–Mac Lane construction; spectra $KO$ and $KU$, representing real and
complex $K$-theory, have $E_{\infty}$-algebra structures whose origin is the tensor product of vector bundles; bordism spectra like $MO$, $MSO$, $MU$, and the like have $E_{\infty}$-algebra structures whose origin is the product structure on manifolds; if $Y$ is an infinite loop space, then there is a spherical group algebra $\mathbb{S}[Y] = \Sigma_{\infty} Y$ with an $E_{\infty}$-algebra structure; and if $R$ is an $E_{\infty}$-algebra and $X$ is a space, there is a spectrum $R^X$ (playing the role of "cochains on $X$ with coefficients in $R$") with an $E_{\infty}$-algebra structure that combines the multiplication on $R$ with the diagonal map $X \to X \times X$.

Problem 1.0.1 dates back to the first systematic studies of $E_{\infty}$-algebras. Understanding why this result is so desirable requires knowing a little about what the Brown–Peterson spectrum is and how important it is in stable homotopy theory.

The complex bordism spectrum $MU$ has an $E_{\infty}$-algebra structure and it is central to Quillen’s relation between stable homotopy theory and formal group laws [Qui69], which initiated the subject of chromatic homotopy theory. However, while almost the entirety of chromatic theory is possible to phrase in terms of $MU$, the $p$-localization $MU(p)$ decomposes into summands equivalent to this irreducible Brown–Peterson spectrum $BP$. The Brown–Peterson spectrum has simpler cohomology and homotopy groups than $MU$ and has canonical descriptions that are internal to the stable homotopy category [Pri80]. The Brown–Peterson spectrum also exhibits the close connection between $p$-local stable homotopy theory and the theory of formal group laws, but with the added benefit that nearly every deep structural property of chromatic homotopy theory or formal group law theory is made more concise and more conceptually accessible through the eyes of $BP$-theory (see, for example, [Rav86] for extensive applications).

The existence of an $E_{\infty}$-algebra structure on $BP$ would be useful in several ways.

- The Adams–Novikov spectral sequence is a method for calculating the set of homotopy classes of maps between two spectra $X$ and $Y$ and can be derived from either their $MU$-homology or $BP$-homology. The computational tools using $MU$-theory (such as the cobar complex) are well behaved with respect to the geometric power operations discussed earlier, which appear in places such as the construction of manifolds of Kervaire invariant one [Bru01]. If $BP$ had an $E_{\infty}$-algebra structure then computations of these geometric power operations using $MU$-theory could instead be related to simpler computations in $BP$-theory.

- Such a structure would allow more concise constructions of many important objects in chromatic theory, such as the Morava $K$-theories $K(n)$ and the truncated Brown–Peterson spectra $BP(n)$, as $BP$-algebras rather than as $MU$-algebras.

- These algebra structures would mean that several computations with these ring spectra could be governed by the computations for $BP$-theory, such as computations of topological Hochschild homology and topological cyclic homology that are important to current work in algebraic $K$-theory [AR02]. These can also be extended by relative computations in $BP$-modules, which are much simpler than the relative calculations in $MU$-modules.

- Perhaps most importantly, the Brown–Peterson spectrum is one of the most prominent examples of an important homology theory where our knowledge of
geometric interpretations (e.g. via Baas–Sullivan theory [Baa73]) lags far behind our algebraic knowledge.\(^1\) Many of the prominent examples of \(E_\infty\)-algebras, such as \(K\)-theory and bordism theory, originate in cohomology theories with geometric cycles or cocycles that have a product. The existence of an \(E_\infty\)-algebra structure on \(BP\) would be a good indicator that a strong geometric interpretation existed.

This problem has generated a great deal of interesting research. The existence of multiplication structures in the homotopy category has a long history (for example, see the introduction of [Str99]). Several forms of obstruction theory have been developed which showed that many spectra constructed by Baas–Sullivan theory admit \(E_1\)-algebra structures [Rob89, Laz01, BJ02, Ang08]. More sophisticated obstruction theory has appeared for \(E_\infty\)-algebras [Rob03, GH04], and Richter obtained lower bounds on the amount of commutativity present in \(BP\) based on Robinson’s obstruction theory [Ric06]. Techniques such as localization and idempotent splitting were developed in [May01] to handle additive and multiplicative versions of the construction of \(BP\). More recently Basterra–Mandell showed that \(BP\) is a split summand of \(MU_\langle p \rangle\) as an \(E_4\)-algebra [BM13], and so the homotopy category of \(BP\)-modules has a symmetric monoidal structure; Chadwick–Mandell used idempotent splittings to show that this could be done with the Quillen idempotent as \(E_2\)-algebras [CM15]. Both Hu–Kriz–May [HKM01] and Baker [Bak14] gave iterative constructions by methods that kill torsion, producing two different types of closest possible torsion-free \(E_\infty\)-algebras to \(BP\). An unpublished paper of Kriz attempted to prove that \(BP\) admits an \(E_\infty\)-algebra structure, and developed the theory of topological André-Quillen (TAQ) cohomology in the process—this theory allows the construction of \(E_\infty\)-algebras by systematically lifting the \(k\)-invariants in the Postnikov tower from ordinary cohomology to TAQ-cohomology [Kri95, Bas99]. Kriz’s original program foundered on a technical detail, but TAQ has been central in a great deal of research since.

However, the hope that Problem 1.0.1 has a positive solution perhaps originated in a time of much greater optimism, and the intervening years have shown that the additive and multiplicative structure of a spectrum are difficult to untangle from each other. Indeed, there is something closer to a reciprocity relationship, where requirements of the additive structure are rewarded with constraints on the multiplicative and vice versa. In line with this, there have been several more recent calculations showing that desirable properties of a multiplication on \(BP\) cannot be realized. Hu–Kriz–May showed that there cannot be a map of \(E_\infty\)-algebras \(BP \to MU_\langle p \rangle\) because it conflicts with calculation of Dyer–Lashof operations in their homology, despite the presence of the Quillen idempotent which describes such a splitting additively and algebraically [HKM01]. In the reverse direction, Johnson–Noel showed with hard calculations that the particular map of ring spectra \(MU_\langle p \rangle \to BP\) employed to great effect in chromatic theory cannot be a map of \(E_\infty\)-algebras for \(p \leq 13\) [JN10], based on a power operation criterion due to McClure [BMMS86, VIII.7.7, 7.8].

The Hu–Kriz–May result seems more decisive mainly because it uses structure that is forced. The mod-\(p\) homology groups \(H, BP\) are identified as a canonical subalgebra

\(^1\)From [May75]: “The point here is that the notion of an \(E_\infty\) ring spectrum seems not to be a purely homotopical one; good concrete geometric models are required, and no such model is known for \(BP\).”
of the dual Steenrod algebra $H_*H_{F_2}$, and this means that the ring structure on $H_*BP$ and operations coming from any $E_n$-algebra structure (including the Dyer–Lashof operations mentioned above) are completely determined by those in the dual Steenrod algebra. It is straightforward to show that $H_*BP$ is closed in $H_*H_{F_2}$ under the Dyer–Lashof operations, and so we cannot exclude the possibility that $BP$ is an $E_\infty$-algebra using a relation between these primary operations. This paper shows that, at the prime 2, there does exist a contradiction for a more subtle reason: while $H_*BP$ is closed under primary operations, it is not closed under secondary operations. This parallels Adams’ solution of the Hopf invariant one problem using secondary cohomology operations [Ada66]. The proof will critically rely on Johnson–Noel’s calculation of power operations in complex bordism.

**Theorem 1.1.1 (5.4.2, 5.4.5).** There exists a natural secondary operation in the mod-2 homology of $E_{12}$-algebras with the following properties. For $R$ an $E_{12}$-algebra, the secondary operation is defined on the subset of $H_*R$ satisfying certain identities between Dyer–Lashof operations in $H_*R$ for $5 \leq k \leq 13$, and the secondary operation takes values in a quotient of $H_*R$. This operation is preserved by maps $R \to R'$ of $E_{12}$-algebras.

In the dual Steenrod algebra $H_*H_{F_2} \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots ]$, this operation is defined on the element $\xi_1^2 \in H_2(H_{F_2})$ and, mod decomposables, unambiguously takes the value $\xi_5 \in H_{31}(H_{F_2})$.

With this theorem, we can exclude the existence of an $E_\infty$-algebra structure on the 2-local Brown–Peterson spectrum and several related objects (e.g. the generalized $BP\langle k \rangle$ whose cohomology is discussed in [LN14, 4.3] and whose additive uniqueness is discussed in [AL17]).

**Theorem 1.1.2 (5.4.6, 5.5.4).** Suppose that $R$ is a connective $E_{12}$-algebra with a ring homomorphism $\pi_0 R \to \mathbb{F}_2$ such that the induced map on mod-2 homology $H_*R \to H_*H_{F_2}$ is injective in degrees 5 through 13. If $\xi_1^2$ is in the image of $H_2(R)$, then the element $\xi_5$ is in the image of $H_{31}R$ mod decomposables.

In particular, the 2-local Brown–Peterson spectrum $BP$, the (generalized) truncated Brown–Peterson spectra $BP\langle k \rangle$ for $k \geq 4$, and their 2-adic completions do not admit the structure of $E_n$-algebras for any $12 \leq n \leq \infty$.

### 1.2 Remarks on obstruction theory

The secondary operation we will define is determined by a relation between Dyer–Lashof operations (the full relation is rather large, and is displayed in Proposition 5.4.1). For us, this relation is not obvious; it is not obvious that this particular relation is relevant; and it is not obvious that the resulting secondary operation is calculable. We did not find this relation by trial and error—or, more accurately, we tried to find relevant secondary operations by trial and error and failed. All of our preliminary attempts resulted in combinations that were excluded by necessity of compatibility with the Steenrod operations. In this section we will indicate a little about how the main result of this paper was found, as opposed to how it is written.

The obstruction theory of Goerss–Hopkins [GH] requires a simplicial operad, an appropriate homology theory $E_\ast$, and an algebra $A$ for this simplicial operad in
$E,E$-comodules. From this, it produces an obstruction theory to calculate the moduli space of algebras over the geometric realization of this operad whose $E$-homology is $A$. Senger specialized this to the case where $E$ is mod-$p$ homology and the operad is a constant $E_{\infty}$-operad [Sen]. His work produced an obstruction theory whose input is an algebra $A$ with Steenrod operations and Dyer–Lashof operations satisfying instability relations and Nishida relations, whose obstruction groups are Ext-groups in this category, and which calculated the moduli space of $E_{\infty}$-algebras whose mod-$p$ homology is $A$. He also developed several tools for reducing these calculations to more tractable Ext-groups that could, in the case of $BP$ or $BP(n)$, be calculated with a Koszul resolution [Pri70]. By construction, this obstruction theory remembers that the Nishida relations will exclude a number of possible obstructions. (The problem encountered in Kriz’s preprint [Kri95] could be viewed as the accidental exclusion of too many obstructions in this fashion.)

In the case of the 2-primary Brown–Peterson spectrum, calculations with this obstruction theory indicated two first potential nonzero obstruction classes. We can define $y$, $R^n$, and $v_m$ to be, respectively, Ext-Koszul dual to the generator $\xi_1^2 \in H_2(BP)$, the Dyer–Lashof operation $Q^{n-1}$, and the Milnor primitive $Q_{m-1}$ in the Steenrod algebra. (The $R^n$ are closely related to unpublished work of Basterra–Mandell on operations in TAQ-cohomology.) Then, using this notation, the first possible obstruction classes are $v_2 R^{19} R^y$ and $v_4 R^{21} R^y$. Under the yoga of secondary operations described by Adams [Ada66], the potential obstruction class $v_4 R^{21} R^y$ would detect an obstruction from a secondary operation whose value involved $\xi_5$ (detected by the Milnor primitive), combining relations that (at least) involved the Adem relations for $Q^{20} Q^8$ and an identity satisfied by $Q^8 \xi_1^2$.

Indeed, our main result is that this is the case. However, much of the progress in this paper traces its origin back to the actual calculation of these relations. After determining the needed identities in Proposition 5.4.1 we could identify most of the relations in $H\ast BP \subset H\ast H\mathbb{F}_2$ as already holding true in $H\ast MU$, making it possible to begin juggling this secondary operation through much simpler ones passing through $H\ast MU$.

1.3 Further questions

We strongly believe that the 2-primary Brown–Peterson spectrum is not unique in failing to admit an $E_{\infty}$-algebra structure.

**Conjecture 1.3.1.** For any odd prime $p$, the $p$-local Brown–Peterson spectrum does not admit the structure of an $E_{\infty}$-algebra.

It seems likely that a similar obstruction using secondary operations could be found, but the author does not know a potential location.

Our initial stepping stone in this paper is a calculation of a Dyer–Lashof operation in the 2-primary dual Steenrod algebra for $MU$-modules.

**Problem 1.3.2.** Determine how the Dyer–Lashof operations act on the $p$-primary $MU$-dual Steenrod algebras $\pi_\ast(H\mathbb{F}_p \wedge_{MU} H\mathbb{F}_p)$.
Baker has shown in [Bak15] how to derive the Nishida relations, describing the interaction between cohomology operations and Dyer–Lashof operations, from the Dyer–Lashof operations in the ordinary dual Steenrod algebra. This suggests that a solution to the previous problem would give additional constraints on $MU$-algebras by describing additional relations that have to hold in their mod-$p$ homology relative to $MU$.

**Problem 1.3.3.** Determine analogues of Nishida relations between the homology operations on $H_*^{MU} R = \pi_* (H^2_* R \wedge_{MU} R)$ and the Dyer–Lashof operations.

In particular, Remark 4.4.4 describes how the Dyer–Lashof operation that we have calculated seem to place a cap on multiplicative structure for the map $MU \to BP$ at the prime 2—a stronger cap than the one we have shown for the amount of multiplicative structure on $BP$.

**Problem 1.3.4.** Find constraints on the values of $n$ for which the $p$-local Brown–Peterson spectrum can admit the structure of an $E_n$ $MU$-algebra.

The calculations of this paper deduce our unexpected Dyer–Lashof operation in the $MU$-dual Steenrod algebra from a multiplicative Dyer–Lashof operation in the Hopf ring for $MU$, and an induced operation in the homology of the space $SL_1(MU) \subset GL_1(MU)$ of strict units. This is a first step towards determining the homology of the spectrum $gl_1(MU)$, about which very little is known, using the Miller spectral sequence [Mil78].

**Problem 1.3.5.** Determine multiplicative Dyer–Lashof operations in the Hopf ring for $MU$ and in the homology of $GL_1(MU)$. Determine homology groups of the unit spectrum $gl_1(MU)$ and the Picard spectrum $\text{pic}(MU)$, as well as information about their homotopy types.

Remark 2.1.9 points out that our description of secondary operations and Toda brackets is not optimal. For example, it sometimes requires strict basepoints for mapping spaces, strict unitality, and strict initial and terminal objects, all of which are not invariant under unbased homotopy equivalences between objects and not invariant under Dwyer–Kan equivalences between topological categories. However, the tools should apply in much wider generality; investigations in this direction have been carried out by Bhattacharya and Hank.

**Problem 1.3.6.** Develop a homotopical framework for secondary operations.

For example, a framework in the language of quasicategories to encode the notion of a category enriched in based spaces would be extremely useful. Ideally, this should make Cohen–Jones–Segal’s construction of filtered spectra from coherent chain complex objects [CJS95, §5] part of an equivalence between filtered objects and coherent chain complex objects in a stable $\infty$-category, extending Lurie’s version of the Dold–Kan correspondence [Lur14, 1.2.4].

Our calculations with power operations in the Hopf ring make use of the $H^2_{\infty}$-algebra structure on $MU$, a concept from [BMMS86] which has been largely neglected in the modern literature. It should be possible to describe a fully coherent version of this structure using the language of Picard spaces and Picard spectra [MS16].
Problem 1.3.7. Give a systematic development of $E_\infty^d$-algebras as homotopy coherent versions of $H_\infty^d$-ring spectra, and show that the $H_\infty^d$-structures on classical Thom spectra constructed in [BMMS86, VIII.5.1] lift to $E_\infty^d$-algebra structures.

An $E_\infty^d$-structure on an $E_\infty$-algebra $R$ should be a lift of the map of spaces
\[ d\mathbb{Z} \subset \mathbb{Z} \subset \text{Pic}(\mathbb{S}) \to \text{Pic}(R) \]
to a map of $E_\infty$-spaces, corresponding to a functor of symmetric monoidal $\infty$-categories. In close analogy with the work of Ando–Blumberg–Gepner–Hopkins–Rezk [ABG+14], the point $\{d\mathbb{K}\} \to \text{Pic}(\mathbb{S})$, whose homotopy colimit should be a Thom spectrum on $B\Sigma_m$ representing the extended power construction on $S^{dk}$. An $E_\infty^d$-structure on $R$ should then make the resulting diagram $B\Sigma_m \to \text{Pic}(R)$ equivalent to a constant diagram with value $S^{dkm}$, allowing us to conclude that the smash product of $R$ with the Thom spectrum has an equivalence to $R \wedge (B\Sigma_m)_+ \wedge S^{dkm}$.

1.4 Outline of proof

We will begin by calculating a Dyer–Lashof operation in the $MU$-dual Steenrod algebra $\pi_*(H \wedge_{MU} H)$, where $H$ is the Eilenberg–Mac Lane spectrum $H\mathbb{F}_2$. This has maps in from the dual Steenrod algebra $\pi_*(H \wedge H)$ and out to the homology of $SU$ which become a left exact sequence
\[ 0 \to Q\pi_*(H \wedge H) \to Q\pi_*(H \wedge_{MU} H) \to QH, SU \]
on indecomposables. The Dyer–Lashof operations on the left are known by work of Steinberger, and were employed by Tilson [Til16] to calculate operations in the middle term. The Dyer–Lashof operations on the right are known by work of Kochman. The operation we will calculate is the first possible hidden extension and it turns out to be nontrivial.

To carry out this calculation we rely on calculations of unstable multiplicative Dyer–Lashof operations. This uses Ravenel–Wilson’s description of Hopf ring structure on the homology of the spaces in the $\Omega$-spectrum for $MU$ [RW74] and a comparison between Dyer–Lashof operations and the tom Dieck–Quillen power operations in $MU$-cohomology. The relevant portion of this extension is ultimately determined by the calculation of Johnson–Noel discussed earlier [JN10]. We will make extensive use of the results of Bruner–May–McClure–Steinberger in doing this calculation [BMMS86].

We will then give an alternative description of this operation in the $MU$-dual Steenrod algebra as a Dyer–Lashof operation applied to the result of a secondary operation. This allows us to use juggling formulas for secondary operations to determine a more complicated secondary operation in the dual Steenrod algebra, showing that $H, BP$ is not closed under secondary operations. (We are fortunate in this regard that most of our calculations can be carried out mod decomposable elements.) In order to work with this we will describe a framework for secondary operations in Section 2 based on Harper’s book [Har02], with our emphasis shifted from suspension and loop operators to loops inside mapping spaces.
1.5 Terminology

The notation Map always denotes a space, or simplicial set, of maps. We will refer to a diagram as homotopy commutative if it commutes in the homotopy category, and homotopy coherent if we have further chosen compatible homotopies and higher homotopies to recover a coherent diagram [Vog73, Lur09].

We will adhere to the standard conventions for function composition and path composition, even though they make no sense. Maps in a category are written using arrows \( X \to Y \), and given \( f: X \to Y \) and \( g: Y \to Z \) there is a composite \( gf \). Paths in a space are written using arrows \( p \Rightarrow q \), and given \( h: p \Rightarrow q \) and \( k: q \Rightarrow r \) there is a path composite \( h \cdot k \).

We will work in the category of symmetric spectra of [HSS00] with the positive stable model structure [MMSS01]. We will use the term commutative ring spectrum for a commutative monoid in symmetric spectra, and the term \( E_n \)-algebra for an algebra over a fixed \( E_n \)-operad in simplicial sets— for this it is convenient to use the \( E_\infty \)-operad of Barratt–Eccles [BE74] with its filtration by \( E_n \)-suboperads due to Berger [Ber97]. In this framework, results of Elmendorf–Mandell imply that each category of \( E_n \)-algebras is a simplicial model category, that the forgetful functors from \( E_n \)-algebras to \( E_m \)-algebras or to symmetric spectra are right Quillen functors, and that there is a Quillen equivalence between \( E_\infty \)-algebras and commutative ring spectra [EM06, 1.3, 1.4].

Throughout this paper, we will write \( H_\ast(X; R) \) for the homology groups of \( X \) with coefficients in \( R \), and similarly for cohomology. If \( R \) is not specified, we view these as being taken with coefficients in a fixed finite field \( \mathbb{F}_p \) of prime order. Homology and cohomology groups of spaces are unreduced unless otherwise specified.

We will let \( MU \) be the complex cobordism spectrum and \( F \) be the formal group law of \( MU \), writing it as \( F(x, y) = x + F \gamma y = \sum a_{i,j}x^i y^j \) with \( a_{i,j} \in \pi_{2(i+j-1)}MU \).

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2 Secondary operations

A secondary composite is the first basic type of obstruction encountered when lifting a homotopy commutative diagram to a homotopy coherent diagram.

**Definition 2.0.1.** Let \( \mathcal{D} \) be a category enriched in spaces. Suppose that we are given the following data:

1. a sequence \((X_0, X_1, X_2, X_3)\) of objects of \( \mathcal{D} \)
2. maps $f_{ij}: X_i \to X_j$ for $i < j$

3. paths $h_{ijk}: f_{jk}f_{ij} \Rightarrow f_{ik}$ in $\text{Map}_D(X_i, X_k)$ for $i < j < k$

Then the associated secondary composite is the element of $\pi_1(\text{Map}_D(X_0, X_3), f_{03})$ represented by the path composite

$$(h_{023})^{-1} \cdot (f_{23}h_{012})^{-1} \cdot (h_{123}f_{01}) \cdot h_{013},$$

viewed as a loop based at $f_{03}$.

In the following sections we will describe secondary composites which are comparable with Massey products or Toda brackets; they rely on the existence of distinguished “null” maps so that we can make sense of composites being trivial. Our perspective is based Harper’s book [Har02].

### 2.1 Secondary operations and brackets

Throughout this section, let $C$ be a category enriched in pointed spaces (or, with appropriate modifications, pointed simplicial sets) under $\wedge$, and write $\text{Map}_C(x, y)$ for the mapping space between any pair of objects of $C$. We refer to the basepoint of this mapping space as the null map or $*$. Null maps satisfy $f* = *f = *$ for any $f$.

**Definition 2.1.1.** Suppose we have maps

$$X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2$$

in $C$. A tethering of this composite is a homotopy class of nullhomotopy of $gf$: a homotopy class of path $h: gf \Rightarrow *$ in $\text{Map}_C(X_0, X_2)$ (cf. [Har02, 4.1.2]). We will write $g \leftrightarrow^h f$ to indicate such a tethering, and $g \leftrightarrow f$ to indicate that there is a chosen tethering which is either implicit or not important to name.

**Remark 2.1.2.** If a triple composite $kgf$ is nullhomotopic, then a tethering $k \leftrightarrow^h gf$ is the same data as a tethering $k \leftrightarrow^h g$. 

**Definition 2.1.3.** Suppose we have maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3,$$

and tetherings $f_{23} \leftrightarrow^h_{123} f_{12} \leftrightarrow^h_{012} f_{01}$. Then we define the element

$$(f_{23} h_{123} f_{12} h_{012} f_{01}) \in \pi_1(\text{Map}_C(X_0, X_3), *)$$

to be the path composite $(f_{23} h_{012})^{-1} \cdot h_{123} f_{01}$ obtained by gluing together the two nullhomotopies $f_{23} f_{12} f_{01} \Rightarrow *$. This is the secondary composite, as in Definition 2.0.1, obtained by choosing $f_{02} = f_{03} = f_{13} = *$ and the trivial nullhomotopies $h_{013}$ and $h_{023}$.
Definition 2.1.4. Suppose we have maps

\[ X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \xrightarrow{f_{23}} X_3. \]

If we have chosen a tethering \( f_{23} \overset{h}{\leftrightarrow} f_{12} \) and \( f_{12}f_{01} \) is nullhomotopic, we write

\[ \langle f_{23} \leftrightarrow f_{12}, f_{01} \rangle \subset \pi_1(\text{Map}_C(X_0, X_3), \ast) \]

for the set of all elements \( \langle f_{23} \overset{h}{\leftrightarrow} f_{12}, k \overset{k}{\leftrightarrow} f_{01} \rangle \) as \( k \) ranges over possible tetherings, and refer to \( \langle f_{23} \overset{h}{\leftrightarrow} f_{12}, \ast \rangle \) as the secondary operation determined by the tethering. The set of maps \( f_{01} \) such that \( f_{12}f_{01} \) is nullhomotopic is referred to as the domain of definition of this secondary operation, and the possibly multivalued nature of this function as the indeterminacy of the secondary operation.

The secondary operations \( \langle \ast, f_{12} \leftrightarrow f_{01} \rangle \) are defined in the same way.

Definition 2.1.5. Suppose we have maps

\[ X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \xrightarrow{f_{23}} X_3 \]

such that the double composites \( f_{23}f_{12} \) and \( f_{12}f_{01} \) are nullhomotopic. We define the subset

\[ \langle f_{23}, f_{12}, f_{01} \rangle \subset \pi_1(\text{Map}_C(X_0, X_3), \ast), \]

or bracket, to be the set of all secondary composites \( \langle f_{23} \leftrightarrow f_{12} \leftrightarrow f_{01} \rangle \).

Proposition 2.1.6. Changing the tethering and homotopy class of maps alters the value of a secondary composite by multiplication by loops, as follows. If \( f_{23} \) is homotopic to \( f'_{23} \), we have

\[ \langle f_{23} \overset{h}{\leftrightarrow} f_{12}, k \overset{k}{\leftrightarrow} f_{01} \rangle = \langle f'_{23} \overset{h'}{\leftrightarrow} f_{12}, k \overset{k}{\leftrightarrow} f_{01} \rangle \cdot (uf_{01}) \]

for some \( u \in \pi_1(\text{Map}_C(X_1, X_3)) \), determined by \( h \) and \( h' \). Similarly, if \( f_{01} \) is homotopic to \( f'_{01} \), we have

\[ \langle f_{23} \overset{h}{\leftrightarrow} f_{12}, k \overset{k}{\leftrightarrow} f_{01} \rangle = \langle f_{23} \overset{h}{\leftrightarrow} f'_{12}, k' \overset{k'}{\leftrightarrow} f_{01} \rangle \]

for some \( v \in \pi_1(\text{Map}_C(X_0, X_2)) \).

If we replace all three maps with homotopic maps and choose new tetherings, we have

\[ \langle f_{23} \overset{h}{\leftrightarrow} f_{12}, k \overset{k}{\leftrightarrow} f_{01} \rangle = \langle f_{23} \overset{h}{\leftrightarrow} f'_{12}, k' \overset{k'}{\leftrightarrow} f'_{01} \rangle \cdot (uf_{01}) \]

for some \( u \in \pi_1(\text{Map}_C(X_1, X_3)) \) and \( v \in \pi_1(\text{Map}_C(X_0, X_2)) \).

In particular, this describes completely the indeterminacy in secondary operations and brackets, and shows that (up to this indeterminacy) a secondary operation or a bracket is well-defined on homotopy classes of maps.
Corollary 2.1.7. A secondary operation \( \langle f_{23} \leftrightarrow h f_{12}, - \rangle \) determines a well-defined map \( \Phi \) on \( \ker f_{12} \subset \pi_0 \Map_C(X_0, X_1) \) whose values are right cosets:

\[
\ker f_{12} \xrightarrow{\Phi} f_{23} \pi_1 \Map_C(X_0, X_2) \backslash \pi_1 \Map_C(X_0, X_3).
\]

If two tetherings \( h, h' \) give rise to operations \( \Phi, \Phi' \), then there exists an element \( u \in \pi_1 \Map_C(X_1, X_3) \) such that

\[
\Phi x = \Phi' x \cdot (ux)
\]

for all \( x \in \ker f_{12} \subset \pi_0 \Map_C(X_0, X_1) \).

Dual results hold for \( \langle - , f_{12} \leftrightarrow f_{01} \rangle \).

Corollary 2.1.8. Suppose we have maps

\[
X_0 \xrightarrow{f_{01}} X_1 \xrightarrow{f_{12}} X_2 \xrightarrow{f_{23}} X_3
\]

such that the double composites \( f_{23} f_{12} \) and \( f_{12} f_{01} \) are nullhomotopic. Then the bracket \( \langle f_{23}, f_{12}, f_{01} \rangle \) depends only on the homotopy classes of \( f_{i,i+1} \) and is a well-defined double coset in

\[
(f_{23} \pi_1 \Map_C(X_0, X_2)) \backslash \pi_1 \Map_C(X_0, X_3) / (\pi_1 \Map_C(X_1, X_3) f_{01}).
\]

Remark 2.1.9. A more flexible version of the above constructions should exist, where basepoints are replaced by some appropriate system of maps \( E_{i,j} \rightarrow \Map_C(X_i, X_j) \) from contractible spaces \( E_{i,j} \), together with appropriate lifts of the composition maps. For example, the category of diagrams of spaces \( E \rightarrow X \) is a monoidal category under the pushout-product, and so we could ask for \( C \) to be enriched in this category with the constraint that the space \( E \) is always contractible. We might instead try to find an appropriate analogue in terms of quasicategories satisfying certain basepoint conditions: in the notation of [Lur09], the data to describe a secondary composite in Definition 2.0.1 defines a map of enriched categories \( \mathbb{C} [\partial \Delta^3] \rightarrow \mathcal{D} \), and the secondary composite is the obstruction to extending it to a map \( \mathbb{C} [\Delta^3] \rightarrow \mathcal{D} \) (a homotopy coherent triple composite). Both of these constructions would apply more widely, but involve more bookkeeping and possibly require a more advanced technical framework. We have elected to use constructions in categories where this will not be necessary in order to minimize the technical load.

The definitions of secondary operations and brackets are preserved in an obvious way under functors between enriched categories.

Proposition 2.1.10. Suppose \( F : C \rightarrow C' \) is an enriched functor between categories enriched in pointed spaces. Then any tethering \( g \leftrightarrow f \) in \( C \) induces a tethering \( Fg \leftrightarrow Ff \) in \( C' \). We have an equality

\[
F(\langle f_{23} \leftrightarrow h f_{12}, \cdots \rangle) = \langle Ff_{23} \leftrightarrow Ff_{12}, \cdots \rangle.
\]
and we have containments as follows:

\[ F(⟨f_{23} h \leftrightarrow f_{12}, f_{01}⟩) \subset ⟨F f_{12}, F f_{01}⟩ \]

\[ F(⟨f_{23}, f_{12} k \leftrightarrow f_{01}⟩) \subset ⟨F f_{12} k \leftrightarrow F f_{01}⟩ \]

\[ F(⟨f_{23}, f_{12}, f_{01}⟩) \subset ⟨F f_{12}, F f_{01}⟩ \]

2.2 Pointings and augmentations

In this section we let \( D \) be a category enriched in spaces (now assumed to have no basepoint). In this section we indicate a construction that replaces \( D \) with a category enriched in pointed spaces.

**Definition 2.2.1.** An augmented object of \( D \) is an object \( X \in D \) equipped with a map \( Y \to ∅ \) to an initial object of \( D \). The space of maps between two augmented objects is the subspace of ordinary maps that commute with the augmentations.

A pointed object of \( D \) is an object \( Z \in D \) equipped with a map \( ∗ \to Z \) from a terminal object of \( D \). The space of maps between two pointed objects is the subspace of ordinary maps that commute with the pointings.

**Definition 2.2.2.** Suppose \( D \) is a category enriched in spaces. We define \( D^\pm \), the category of possibly pointed or augmented objects of \( D \), to be the following category enriched in based spaces.

An object of \( D^\pm \) is one of three types:

1. an augmented object \( X \to ∅ \) of \( D \),
2. an ordinary object \( Y \) of \( D \), or
3. a pointed object \( ∗ \to Z \) of \( D \).

The mapping spaces in \( D^\pm \) are given as follows.

1. The space of maps between two augmented objects \( X \to ∅, X' \to ∅' \) is the space of maps of augmented objects, with basepoint given by the composite \( X \to ∅ \to X' \).
2. The space of maps between two pointed objects \( ∗ \to Z, ∗' \to Z' \) is the space of maps of pointed objects, with basepoint given by the composite \( Z \to ∗' \to Z' \).
3. The space of maps between two ordinary objects \( Y, Y' \) is the based space \( \text{Map}_D(Y, Y')_+ \), whose disjoint basepoint is called the formal null map.
4. The space of maps from an augmented object \( X \to ∅ \) to an ordinary object \( Y \) is the space of maps \( X \to Y \), with basepoint given by the map \( X \to ∅ \to Y \).
5. The space of maps from an ordinary object \( Y \) to a pointed object \( ∗ \to Z \) is the space of maps \( Y \to Z \), with basepoint given by the map \( Y \to ∗ \to Z \).
6. The space of maps from an augmented object $X \to \emptyset$ to a pointed object $* \to Z$ is the space of maps $X \to Z$, with basepoint given by the canonical map factoring through either $\emptyset$ or $*$ in the commutative diagram

$$
\begin{array}{ccc}
X & \to & \emptyset \\
\downarrow & & \downarrow \\
* & \to & Z
\end{array}
$$

7. All other mapping spaces are one-point spaces—there are no non-basepoint maps from ordinary objects to augmented ones, or from pointed objects to ordinary ones. We also refer to these as formal null maps.

We have full subcategories of $\mathcal{D}^\pm$ spanned by fewer than all three of these types of objects: for example, we have the categories of augmented objects, pointed objects, possibly augmented objects, and possibly pointed objects of $\mathcal{D}$.

**Proposition 2.2.3.** The category $\mathcal{D}^\pm$ is enriched in pointed spaces under $\wedge$.

In $\mathcal{D}^\pm$, if a composite $X \to Y \to Z$ is nullhomotopic then $X$ is augmented, $Z$ is pointed, or one of the maps is a formal null map (in which case there is a canonical tethering).

This construction makes it possible to take a category $\mathcal{D}$ and sensibly talk about secondary operations and brackets for a composite $X_0 \to X_1 \to X_2 \to X_3$ in $\mathcal{D}$ if the first map is a map of augmented objects, if the last map is a map of pointed objects, or if the first object is augmented and the last object is pointed. (If the maps arise from $\mathcal{D}$ then a formal null map cannot appear.)

**Example 2.2.4.** If $C$ has homotopy pushouts and we have augmented objects $X_0 \to X_1 \to \emptyset$, the bracket can be identified with an element in $\pi_1 \Map_C(\Sigma X_0, X_3)$, represented by the outside rectangle in the homotopy coherent diagram

$$
\begin{array}{ccc}
X_0 & \to & X_1 & \to & \emptyset \\
\downarrow & & \downarrow & & \downarrow \\
\emptyset & \to & X_2 & \to & X_3
\end{array}
$$

The indeterminacy in the bracket is given by path concatenation with composites of either of the following forms:

$$
\Sigma X_0 \xrightarrow{\nu} X_2 \xrightarrow{f_{03}} X_3 \quad \Sigma X_0 \xrightarrow{\Sigma f_{03}} \Sigma X_1 \xrightarrow{u} X_3
$$

Dual results hold if we are given pointed objects $* \to X_2 \to X_3$, so that the bracket can be identified with an element in $\pi_1 \Map_C(X_0, \Omega X_3)$. To avoid grief in these identifications, especially with respect to a loop-suspension adjunction, it is important to pay attention to the orientation of $S^1$ as detailed at length in [Har02]. This is why we have indicated directions for 2-cells.
In the “mixed” case, there is little profound that we can say other than identification of a element in the bracket with the loop determined by a homotopy coherent diagram

\[
\begin{array}{c}
X_0 \rightarrow X_1 \rightarrow \ast \\
\downarrow \quad \downarrow \\
0 \rightarrow X_2 \rightarrow X_3.
\end{array}
\]

### 2.3 Juggling and Peterson–Stein formulas

In this section we return to assuming that we have a category \(C\) enriched in based spaces.

There are several “juggling” formulas that describe the relationship between brackets and function composition. All of them are obtained by choosing representative nullhomotopies and composing them appropriately, as in the Peterson–Stein formulas \([PS59]\).

**Lemma 2.3.1.** Suppose we have a sequence of objects \((X_0, \ldots, X_4)\), together with maps \(f_{i, i+1} : X_i \rightarrow X_{i+1}\) and tetherings

\[
f_{34} \leftrightarrow f_{23} \leftrightarrow f_{12} \leftrightarrow f_{01}.
\]

Then there is an identity

\[
f_{34}(f_{23} \leftrightarrow f_{12} \leftrightarrow f_{01})^{-1} = (f_{34} \leftrightarrow f_{23} \leftrightarrow f_{12})f_{01}
\]

in \(\pi_1 \text{Map}_C(X_0, X_4)\).

**Example 2.3.2.** In the case where \(X_2 \rightarrow X_3 \rightarrow X_4\) are maps of pointed objects in a category \(D\), this Peterson–Stein relation expresses that both loops in \(\text{Map}_D(X_0, X_3)\) are homotopic to the loop determined by the following homotopy coherent diagram:

\[
\begin{array}{c}
X_0 \rightarrow \ast \\
\downarrow \quad \downarrow \\
X_1 \rightarrow X_2 \rightarrow \ast \\
\downarrow \quad \downarrow \\
\ast \rightarrow X_3 \rightarrow X_4
\end{array}
\]

Similarly, in the mixed case we will need to derive Peterson–Stein relations from diagrams such as the following:

\[
\begin{array}{c}
X_0 \rightarrow 0 \\
\downarrow \quad \downarrow \\
X_1 \rightarrow X_2 \rightarrow \ast \\
\downarrow \quad \downarrow \\
0 \rightarrow X_3 \rightarrow X_4
\end{array}
\]
Lemma 2.3.3. Each of the following juggling formulas holds whenever defined.

\[
\langle f_4 \leftrightarrow f_3, f_2, f_1 \rangle = f_4(f_3, f_2, f_1)^{-1} \\
\langle f_3, f_2, f_1 \rangle = \langle f_4 f_5 f_6, f_2, f_1 \rangle \\
\langle f_4 f_3 f_2, f_1 \rangle \supset \langle f_4 f_3 f_2, f_1 \rangle \\
\langle f_4 f_2, f_1 \rangle \supset \langle f_4 f_3 f_2, f_1 \rangle \\
\langle f_4 f_3 f_2 f_1 \rangle \supset \langle f_4 f_3 f_2 f_1 \rangle
\]

Dual results hold for secondary operations \( \langle -, f_2 \leftrightarrow f_1 \rangle \).

Proposition 2.3.4. Each of the following juggling formulas for secondary operations holds whenever both sides are defined:

\[
\langle f_4, f_3, f_2 \rangle f_1 = f_4(f_3, f_2, f_1)^{-1} \\
f_4(f_3, f_2, f_1) = \langle f_4 f_5 f_6, f_2, f_1 \rangle \\
\langle f_4 f_3, f_2, f_1 \rangle \supset \langle f_4 f_3 f_2, f_1 \rangle \\
\langle f_4 f_2, f_1 \rangle \supset \langle f_4 f_3 f_2, f_1 \rangle \\
\langle f_4, f_3 f_2 f_1 \rangle \supset \langle f_4, f_3 f_2 f_1 \rangle
\]

We end with a remark on adjunctions. In the presence of an (enriched) adjunction between categories \( C \) and \( D \), we can describe relationships between secondary operations. Recall that an enriched functor \( F: C \to D \) with enriched left adjoint \( G \) determines (and is determined by) an enriched category \( \mathcal{E} \) with object set \( \text{Ob}(C) \cup \text{Ob}(D) \), such that:

\[
\text{Map}_\mathcal{E}(x, y) = \begin{cases} 
\text{Map}_C(x, y) & \text{if } x, y \in C \\
\text{Map}_D(x, y) & \text{if } x, y \in \mathcal{D} \\
\text{Map}_C(x, Gy) = \text{Map}_D(Fx, y) & \text{if } x \in C, y \in \mathcal{D} \\
\emptyset & \text{otherwise}
\end{cases}
\]
This allows us to describe augmented and pointed objects in the presence of an adjunction and define brackets even amongst objects in categories related by adjunctions. We could, if desired, rephrase several of our constructions in these terms, in particular with respect to brackets that involve maps out of free objects.

### 2.4 Additive structures

In some prominent examples, some of the mapping spaces in \( C \) have natural “addition” structures.

**Definition 2.4.1.** An object \( Y \in C \) is an \( H \)-object if \( \text{Map}_C(\_, Y) \) naturally takes values in \( H \)-spaces: it is equipped with a natural homotopy-unital binary operation \(+\) whose unit is the basepoint. A map of \( H \)-objects is a map \( Y \to Y' \) preserving this structure.

An object \( X \in C \) is an \( \text{co}-H \)-object if \( \text{Map}_C(X, \_) \) naturally takes values in \( H \)-spaces: it is equipped with a natural homotopy-unital binary operation \(+\) whose unit is the basepoint. A map of \( \text{co}-H \)-objects is a map \( X \to X' \) preserving this structure.

**Proposition 2.4.2.** Suppose \( X \) is a \( \text{co}-H \)-object in \( C \) and that we have maps \( f, f' : X \to Y \) and \( g : Y \to Z \), together with tetherings \( h \leftrightarrow f \) and \( g \leftrightarrow f' \). Then the pointwise product on paths in \( \text{Map}_C(X, \_) \) gives a tethering \( h \leftrightarrow h' \leftrightarrow (f + f') \).

**Proposition 2.4.3.** Each of the following addition formulas holds whenever both sides are defined and the source object is an \( \text{co}-H \)-object in \( C \):

\[
\langle f_3 \leftrightarrow h f_2 \leftrightarrow k f_1 \rangle + \langle f_3 \leftrightarrow h f_2 \leftrightarrow f_1' \rangle = \langle f_3 \leftrightarrow h f_2 \leftrightarrow f_1 + f_1' \rangle
\]

\[
\langle f_3 \leftrightarrow h f_2, (f_1 + f_1') \rangle < \langle f_3 \leftrightarrow h f_2, f_1 \rangle + \langle f_3 \leftrightarrow h f_2, f_1' \rangle
\]

Dual results hold for \( H \)-objects.

Here the addition in \( \pi_1 \text{Map}_C(X_0, X_3) \) is, by the Eckmann–Hilton argument, equivalent to either path concatenation or the pointwise \( H \)-space structure on paths.

### 2.5 Secondary power operations

We now specialize the previous discussion to the category of \( E_n \)-algebras over the mod-2 Eilenberg–Mac Lane spectrum \( H \).

**Definition 2.5.1.** Given a commutative ring spectrum \( A \), we let \( P^E_A \) be the left adjoint to the forgetful functor from \( E_n \)-\( A \)-algebras to spectra; if \( n = \infty \) we simply write \( P_A \).

In particular, there is an isomorphism

\[
P^E_A(X) \cong \bigvee A \wedge E_n(k)_{\Sigma_k} \wedge X^k
\]

where the spaces \( E_n(k) \) are the terms in our chosen \( E_n \)-operad, and the set of homotopy classes of maps of \( E_n \)-\( A \)-algebras \( P^E_A(\_ \wedge k) \to C \) is naturally isomorphic to \( \prod \pi_k C \).

The natural map \( X \to \* \) becomes a natural augmentation \( P^E_A(X) \to A \), and a pinch map \( X \to X \vee X \) gives \( P^E_A(X) \) the structure of a \( \text{co}-H \)-object.
**Definition 2.5.2.** A homotopy operation on $E_n A$-algebras is a natural transformation of functors

$$\prod \pi_k (-) \to \pi_j (-),$$

represented by a homotopy class of map of $E_n A$-algebras

$$\mathbb{P}^E_A(S^j) \to \mathbb{P}^E_A(\vee S^k_i)$$

or equivalently an element of

$$\pi_j \mathbb{P}^E_A(\vee S^k_i) \cong \pi_j (A \land \mathbb{P}^E_A(\vee S^k_i)).$$

If this operation preserves the zero element, we view it as determined by a map of augmented objects via the canonical projection to $A$; if it preserves addition, we view it as determined by a map of co-H-objects.

Similarly, if $B$ is an $E_n A$-algebra, a homotopy operation on $E_n A$-algebras under $B$ is a natural transformation in the homotopy category of $E_n A$-algebras under $B$, represented by a homotopy class of map

$$B \amalg \mathbb{P}^E_A(S^j) \to B \amalg \mathbb{P}^E_A(\vee S^k_i).$$

Here the coproduct $\amalg$ takes place in the category of $E_n$-algebras. If this operation preserves the zero element, we view it as determined by a map of augmented objects via the canonical projection to $B$; if it preserves addition, we view it as determined by a map of co-H-objects.

Taking $B = A$ shows that the first type of operations are a special case of the second, so there is no loss of generality in restricting our attention to operations in the relative case. Conversely, if $n = \infty$ then $E_\infty A$-algebras under $B$ are equivalent to $E_\infty B$-algebras.

**Example 2.5.3.** For any $b \in \pi_k(B)$ and any $n > 0$, multiplication by $b$ determines an additive homotopy operation on $E_n A$-algebras under $B$.

**Remark 2.5.4.** As above, the Yoneda lemma allows homotopy operations to be expressed as pre-composition with maps of free algebras. We usually write precomposition on the right, but this is at odds with the standard convention of writing operators (such as the Dyer–Lashof operations) on the left. We could attempt to solve this in many ways. One would be to work in an opposite category so that function application is on the right. One would be to notationally distinguish between maps between free algebras (operations), maps from free algebras to ordinary algebras (homotopy elements), and maps between ordinary algebras (maps). One is to accept the state of affairs, and resist the urge to use the same names for a Dyer–Lashof operation $Q^n$ and the map $\mathbb{P}_H(S^{j+n}) \to \mathbb{P}_H(S^j)$ that represents it. None of these solutions are good, but we have adopted the third because (in all honesty) it has confused us the least.

Relations between homotopy operations allow us to define secondary operations in the following way.
**Definition 2.5.5.** Let $A$ be a commutative ring spectrum and $B$ an $E_n$-$A$-algebra. Suppose we have homotopy operations $Q_i: \prod_i \pi_{l_i} \to \pi_{k_i}$ and $R: \prod_i \pi_{k_i} \to \pi_j$ that preserve zero such that $R \circ (\prod_i Q_i) = 0$, realized by a homotopy coherent diagram

\[
\begin{array}{ccc}
B \amalg \mathbb{P}^E_A(S^j) & \xrightarrow{R} & B \amalg \mathbb{P}^E_A(\bigvee_i S^{k_i}) \\
& \Rightarrow & \\
B & \xrightarrow{Q_i} & B \amalg \mathbb{P}^E_A(\bigvee_i S^{l_i, s})
\end{array}
\]

of augmented $E_n$-$A$-algebras under $B$. We refer to $R$ as a relation between the operations $Q_i$. The coherence produces a tethering homotopy $h$, and the secondary operation associated to this relation is $\langle - \overset{h}{\leftrightarrow} Q \rangle$.

**Proposition 2.5.6.** Given $C$ any $E_n$-$A$-algebra under $B$, the domain of definition of the secondary operation $\langle - \overset{h}{\leftrightarrow} Q \rangle$ is the subset of $\prod \pi_{l_i, s} C$ of collections of elements $x_{i, s} \in \pi_{l_i, s} C$ such that $Q_i(x_{i, s}) = 0$ for all $i$. These are represented by homotopy commutative diagrams

\[
\begin{array}{ccc}
B \amalg \mathbb{P}^E_A(\bigvee_i S^{k_i}) & \rightarrow & B \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \space
\end{array}
\]

of $E_n$-$A$-algebras under $B$. The value of $\langle - \overset{h}{\leftrightarrow} Q \rangle$ is a subset of $\pi_{j+1} C$, and the indeterminacy consists of adding elements in the image of the suspended operation $\sigma R: \prod \pi_{k_{j+1}} C \to \pi_{j+1} C$.

**Remark 2.5.7.** If $n < m \leq \infty$, then the forgetful functors from $E_m$-$A$-algebras under $B$ to $E_n$-$A$-algebras under $B$ preserve secondary operations by Proposition 2.1.10.

**Proposition 2.5.8.** Maps $f: C \to D$ of $E_n$-$A$-algebras under $B$ preserve secondary operations.

**Proof.** This is the statement that

\[f(\langle x, Q \overset{h}{\leftrightarrow} R \rangle) \subset \langle f(x), Q \overset{h}{\leftrightarrow} R \rangle,\]

which is an application of the juggling formulas from Proposition 2.3.4. □

We can also define functional homotopy operations as the analogues of Steenrod’s functional cohomology operations.

**Definition 2.5.9.** Suppose $A$ is a commutative ring spectrum and that we have maps $B \to C \to D$ of $E_n$-$A$-algebras, making $f: C \to D$ a map under $B$. Suppose that we
have a homotopy operation $Q : \prod \pi_l \to \pi_k$ for $E_n A$-algebras under $B$ that preserves zero, realized by a commutative diagram

$$
\begin{array}{ccc}
B \amalg \mathbb{P}^E_A(S^k) & \overset{Q}{\longrightarrow} & B \amalg \mathbb{P}^E_A(S^l) \\
\downarrow & & \downarrow \\
B & \overset{x_s}{\longrightarrow} & B \\
\end{array}
$$

The functional homotopy operation associated to this relation is the bracket $\langle f, -, Q \rangle$.

**Proposition 2.5.10.** For any maps of $E_n A$-algebras $B \to C \to D$, the domain of definition of the functional operation $\langle f, -, Q \rangle$ is the subset of $\prod \pi_l C$ of collections of elements $x_s \in \pi_l C$ such that $f(x_s) = 0$ and $Q(x_s) = 0$. These are represented by homotopy commutative diagrams

$$
\begin{array}{ccc}
B \amalg \mathbb{P}^E_A(S^k) & \overset{Q}{\longrightarrow} & B \amalg \mathbb{P}^E_A(S^l) \\
\downarrow & & \downarrow \\
B & \overset{x_s}{\longrightarrow} & C \\
\downarrow & & \downarrow \\
& \overset{f}{\longrightarrow} & D \\
\end{array}
$$

of $E_n A$-algebras. The value of $\langle f, -, Q \rangle$ is a subset of $\pi_{k+1} D$, and the indeterminacy consists of adding elements in the image of the suspended operation $\sigma Q : \prod \pi_{l+1} C \to \pi_{k+1} D$ and elements in the image of $f : \pi_{k+1} C \to \pi_{k+1} D$.

As in Example 2.5.3, multiplication is one classical example of a homotopy operation. Other examples of homotopy operations, and relations between them, are furnished by power operations.

**Theorem 2.5.11 ([BMMS86, III.3]).** For any commutative $H$-algebra $A$, there are homotopy operations

$$Q^s : \pi_k \to \pi_{k+s}$$

for $E_n A$-algebras when $s < k + n - 1$. These satisfy the following relations.

1. The additivity relation: $Q^s(x + y) = Q^s(x) + Q^s(y)$
2. The instability relations: $Q^s x = x^s$ when $|x| = s$, $Q^s x = 0$ when $|x| > s$
3. The Cartan formula: $Q^s(xy) = \sum_{p+q=s} \mathbb{Q}^p(x)Q^q(y)$
4. The Adem relations: If $r > 2s$, $Q^r Q^s (x) = \sum \binom{r+s-1}{2s-1} Q^{r+s-i} Q^i$

For $m \leq n$, the forgetful map from $E_n$-algebras to $E_m$-algebras preserves Dyer–Lashof operations.

**Proposition 2.5.12.** For any commutative $H$-algebra $A$, all homotopy operations for $E_\infty$ $A$-algebras are generated by multiplication by constants in $\pi_* A$, Dyer–Lashof operations, addition, multiplication, and constants.
Proof. The set of homotopy operations $\prod \pi_n \to \pi_*$ in this category is isomorphic to

$$\pi_*(A \land \mathbb{P}(\vee S^k)) \cong A_* \otimes H_* \mathbb{P}(\vee S^k).$$

Therefore, any homotopy operation is a sum of homotopy operations for $H$-algebras multiplied by constants from $\pi_* A$. However, in [BMMS86] it is shown that the homology $H_* \mathbb{P}(X)$ is the free commutative algebra with Dyer–Lashof operations (subject to the additivity formula, instability relations, Cartan formula, and Adem relations) on $H_* X$, and so the homotopy operations for $H$-algebras are generated by constants, addition, multiplication, and the Dyer–Lashof operations $Q^*$. □

The category of $E_n$ $A$-algebras under $B$ has suspensions, and the suspension of the augmented object $B \prod \mathbb{P}^E_A (\vee S^k)$ is $B \prod \mathbb{P}^E_A (\vee S^{k+1})$

**Proposition 2.5.13.** The suspension operator $\sigma$, on homotopy operations for $E_n$ $A$-algebras under $B$, takes zero-preserving homotopy operations $\prod \pi_n \to \pi_k$ to homotopy operations $\prod \pi_{n+1} \to \pi_{k+1}$. Suspension preserves addition, composition, and multiplication by scalars from $B$. Suspension also takes $Q^*$ to $Q^*$ and takes the binary multiplication operation $\pi_p \times \pi_q \to \pi_{p+q}$ to the trivial operation.

**Remark 2.5.14.** For $E_n$ $H$-algebras, there is also a “top” operation $\xi_{n-1}$ which, if $C$ extends to an $E_{n+1}$-algebra, agrees with to $Q^{k+n-1}$ on classes in $\pi_k C$. However, the top operation satisfies less tractable versions of the identities enjoyed by the remaining operations—most prominently, additivity requires correction by a binary operation called the Browder bracket [BMMS86, III.3.3].

### 2.6 Spectra and geometric realization

For the following, we note that a tethering of a composite map of spectra $X \xrightarrow{f} Y \xrightarrow{g} Z$ is equivalent to a homotopy class of extension from the mapping cone $C f$ to $Z$, up to orientation for the interval component of the mapping cone.

**Proposition 2.6.1.** Suppose $X$, $Y$, and $Z$ are spectra, $X \xrightarrow{f} Y \xrightarrow{g} Z$ is nullhomotopic, and that $\alpha \in \ker(f) \subset \pi_n(X)$ is represented by a map $S^n \to X$. Given any extension $Y \xrightarrow{h} C f \xrightarrow{h} Z$ from the mapping cone representing a tethering, the secondary operation $(g \leftrightarrow f, \alpha)$ is (up to sign) the set $h(\partial^{-1} \alpha)$, where $\partial : \pi_{n+1} C f \to \pi_n X$ is the connecting homomorphism in the long exact sequence of homotopy groups.

**Corollary 2.6.2.** Suppose that $X_\bullet$ is a simplicial spectrum with geometric realization $|X_\bullet|$ and that $F$ is the homotopy fiber of the face map $X_1 \xrightarrow{d_0} X_0$. Then the composite $F \xrightarrow{d_1} X_0 \xrightarrow{i} |X_\bullet|$ has a canonical tethering. If $\alpha \in \pi_n(F) \subset \pi_n X_1$ is in the kernel of $d_1$, then in the geometric realization spectral sequence

$$H_p(\pi_q X_\bullet) \Rightarrow \pi_{p+q} |X_\bullet|$$

the secondary operation $(d_1 \leftrightarrow i, \alpha)$ is represented (up to sign) by the element $[\alpha] \in H_1(\pi_n X_\bullet)$ in the spectral sequence.
Proof. The canonical tethering comes from observing that the mapping cone of the map \( d_1 : F \to X_0 \) maps, by a canonical homotopy equivalence, to the 1-skeleton of the geometric realization \(|X_4|\). The result then follows from the fact that the geometric realization spectral sequence arises from the filtration of \(|X_4|\) by skeleta. □

We will now specialize to discuss how certain elements in a Künneth spectral sequence can be identified with the results of secondary operations.

**Proposition 2.6.3.** Suppose \( f : R \to S \) is a map of commutative ring spectra, and let \( i = 1 \land f : S \land R \to S \land S \). Then, in the (pointed) category of augmented commutative \( S \)-algebras, there is a canonical tethering \( p \leftrightarrow i \) for the composite

\[
S \land R \overset{i}{\to} S \land S \overset{p}{\to} S \land S.
\]

Let \( x \in \pi_n(S \land R) \) map to zero in \( \pi_n(S \land S) \), so that \( \sigma x = \langle p \leftrightarrow i, x \rangle \subset \pi_{n+1}(S \land RS) \) is defined. Then \( \sigma x \) is detected by the image of \( x \) under \( \pi_n(S \land R) \to \pi_n(S \land R \land S) \) in the two-sided bar construction spectral sequence

\[
H_p \pi_q(S \land R^* \land S) \Rightarrow \pi_{p+q}(S \land S).
\]

Proof. The relative smash product receives a map from the end of the augmented simplicial bar construction

\[
S \land R \land S \to S \land S \to S \land R S,
\]

a diagram of commutative ring spectra. The face maps

\[
d_j : S \land R \to S \land R \land S \to S \land S
\]

are the null map \( S \land R \to S \eta \to S \land S \) for \( j = 0 \) and the map \( S \land R \overset{i}{\to} S \land S \) for \( j = 1 \). Because the two composites \( S \land R \to S \land R S \) are homotopic, this provides a canonical tethering in the category of \( S \)-algebras.

A homotopy element \( x \) as described comes from a homotopy coherent diagram as follows:

The two lower right-hand squares define the bracket \( \langle p, i, x \rangle \) in augmented commutative \( S \)-algebras, while the outside of the diagram is made up of two large (2-by-2 and 2-by-1) rectangles that are the result of forgetting down to spectra. However, by Corollary 2.6.2 the outside square determines an element of \( \pi_{n+1}(S \land RS) \) which lifts to the desired element in the two-sided bar construction spectral sequence. □
Remark 2.6.4. The tethering plays an important role here. If we do not impose that the tethering $p \leftrightarrow i$ comes from a tethering in $E_\infty$-algebras, rather than spectra, then the indeterminacy for the bracket in spectra is too large to determine anything about bracket in $E_\infty$-algebras.

If $\pi_*(S \wedge R)$ is flat over $\pi_* S$, we can identify the $E_2$-term in the two-sided bar construction spectral sequence:

$$E^2_{*+*} = \text{Tor}^{\pi_*(S \wedge R)}_{**}(\pi_*(S \wedge S), \pi_*) \Rightarrow \pi_*(S \wedge S)$$

The element $x$ gives rise to the corresponding element in $\text{Tor}_{1,n}$. In particular, we have the following result when the target is the mod-2 Eilenberg–Mac Lane spectrum.

**Proposition 2.6.5.** Suppose $R \to H$ is a map of $E_\infty$-algebras and $x \in H_n R$ maps to zero in the dual Steenrod algebra $H H$. Then there is an element $\sigma x = \langle p \leftrightarrow i, x \rangle$ in the $R$-dual Steenrod algebra $\pi_*(H \wedge R H)$ which is detected by the image of $x$ in homological filtration 1 of the spectral sequence

$$\text{Tor}_{1,n}^H R(H_*, H_* H) \Rightarrow \pi_*(H \wedge H R)$$

**Proof.** In this case, we can rectify the map $R \to H$ to a weakly equivalent map between commutative ring spectra and apply Proposition 2.6.3. □

We now specialize this result to the case where $R$ is the complex bordism spectrum.

**Proposition 2.6.6.** Let $n$ be an integer which is not of the form $2^k - 1$ for any $k$, so that the corresponding generator $b_n \in H_{2n}MU \cong \mathbb{F}_2[b_1, b_2, \ldots]$ in mod-2 homology is the Hurewicz image of the generator $x_n \in \pi_{2n}MU \cong \mathbb{Z}[x_1, x_2, \ldots]$. Then the diagram of $E_\infty$ $H$-algebras

$$\mathbb{F}_2(S^{2n}) \xrightarrow{b_n} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge H_{MU},$$

determines a bracket, and $(p, i, b_n) \equiv \sigma x_n \mod$ decomposables.

**Proof.** The map $H_*MU \to H_* H$ is the map

$$\mathbb{F}_2[b_1, b_2, \ldots] \to \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

that sends $b_{2^k-1}$ to $\xi_k^2$ and sends the other generators to zero. In particular, the Künneth spectral sequence

$$\text{Tor}_{1,n}^{H_*MU}(H_*, H_* H) \Rightarrow \pi_*(H \wedge H_{MU})$$

has as $E_2$-term an exterior algebra $\Lambda[\xi_k] \otimes \Lambda[\sigma b_n \mid k \neq 2^k - 1]$. By comparison with the Künneth spectral sequence

$$\text{Tor}_{1,n}^{\pi_*MU}(H_*, H_*) \Rightarrow \pi_*(H \wedge H_{MU})$$

which degenerates and has $E_2$-term $\Lambda[\sigma x_k]$ of the same (graded) dimension, we find that spectral sequence (2.1) degenerates and that $\sigma b_n$ is congruent to $\sigma x_n \mod$ decomposables for $n$ not of the form $2^k - 1$. We can then apply Proposition 2.6.5 to identify $\sigma x_n$ as a secondary operation. □
3 Hopf rings

3.1 Background

In this section we will recall some of the work of Ravenel–Wilson on Hopf rings [RW74].

Let $E$ be a spectrum with a homotopy commutative multiplication and let $\{E_n\}_{n \in \mathbb{Z}}$ be an associated $\Omega$-spectrum. Then for any ring $R$ the homology groups $H_*(E_*, R)$ have the structure of a Hopf ring: they have a coproduct $\Delta$, an additive product $#$, and a multiplicative product $\circ$ satisfying associativity, commutativity, unitality, and distributivity laws that make them into a graded ring object in coalgebras [RW74, 1.12].

The constants $c \in E^n = \pi_0 E_n$ give rise to elements $[c] \in H_0(E_n; R)$ under the Hurewicz map.

Definition 3.1.1. Suppose $E$ has a complex orientation $x \in E^2(\mathbb{C}P^\infty)$ realized by a based map $b : \mathbb{C}P^\infty \to E_2$, and let $\beta_i \in H_2(\mathbb{C}P^\infty; R)$ be dual to the generator $t^i \in H^* (\mathbb{C}P^\infty; R) \cong R[t]$. We define the classes $b_i \in H_2(E_2; R)$ to be the images of $\beta_i$ under $f$.

Theorem 3.1.2 ([RW74, 4.6, 4.15, 4.20]). Let $\{MU_n\}$ be an $\Omega$-spectrum associated to complex cobordism. For any ring $R$ and any $n \in \mathbb{Z}$, $H_*(MU_2n; R)$ is, as an algebra under $\#$, the tensor product of the group algebra $\mathbb{Z}[\pi_{-2n}MU]$ with a polynomial algebra over $R$.

The even-degree indecomposables $Q\# \tilde{H}_*(MU_2*; R)$ under the $\#$-product form a commutative graded ring under $\circ$, with relations as follows. If we define a formal power series $b(s) = \sum b_i s^i$ and write $x + y = \sum a_i x^i y^j$ for the formal group law of $MU_*$, then we have the Ravenel–Wilson relations

$$b(s + t) = \sum [a_{i,j}] \circ b(s)^i \circ b(t)^j.$$ (3.1)

The ring $Q\# \tilde{H}_*(MU_2*; R)$ is a quotient of the graded ring $R[b_1] \otimes MU^{-2*}$ by a regular sequence, determined by the Ravenel–Wilson relations. Both $Q\# \tilde{H}_*(MU_2*; R)$ and $H_*(MU_2*; R)$ are free over $R$.

Corollary 3.1.3. For all $n$ and all primes $p$, we have commutative diagrams of the following form:

$$
\begin{array}{cccc}
H_*(MU_{2n}; \mathbb{Q}) & \leftrightarrow & H_*(MU_{2n}; \mathbb{Z}) & \rightarrow & H_*(MU_{2n}) \\
\downarrow & & \downarrow & & \downarrow \\
H_{-2n}(MU; \mathbb{Q}) & \leftrightarrow & H_{-2n}(MU; \mathbb{Z}) & \rightarrow & H_{-2n}(MU)
\end{array}
$$

Ravenel–Wilson write $x * y$ for the additive product and $x \circ y$ for the multiplicative product, while Cohen–Lada–May [CLM76] write $xy$ for the additive product and $x \# y$ for the multiplicative product.
3.2 The unstable homology invariant

In the following, for spaces $X$ and $Y$ we will find it convenient to identify $\text{Hom}(H_*X, H_*Y)$ with the isomorphic completed tensor product

$$H_*(Y) \hat{\otimes} H^*(X).$$

Here $H_*(Y)$ is discrete, while $H^*(X)$ inherits an inverse limit structure dual to the filtration of $H_*(X)$ by finite-dimensional subspaces.

The invariant below, in a slightly different form, appears as the “total unstable operation” in [Goe99, 10.2] and is credited to Strickland.

**Definition 3.2.1.** Let $E$ be a multiplicative generalized cohomology theory represented by an $\Omega$-spectrum $\{ E_n \}_{n \in \mathbb{Z}}$. The **unstable homology invariant** for $E$-cohomology is the collection of natural transformations of sets $\Lambda: E^n(X) = [X, E_n] \to \text{Hom}(H_*X, H_*E_n) \cong H_*E_n \hat{\otimes} H^*(X)$.

**Remark 3.2.2.** For any $a$, the element $\Lambda(a) \in \text{Hom}(H_*X, H_*E_n)$ is a coalgebra map that respects the Steenrod operations. This restriction will not be necessary for us to take into account here.

The groups $H_*E_n \hat{\otimes} H^*(X)$ have products $\#$ and $\circ$, each individually induced by the corresponding product in the Hopf ring and the cup product in $H^*(X)$. Using these, we can determine how $\Lambda$ interacts with the ring structure in $E$-cohomology.

**Proposition 3.2.3.** The unstable homology invariant $\Lambda$ satisfies the following formulas:

$$\Lambda(x + y) = \Lambda(x) \# \Lambda(y)$$
$$\Lambda(xy) = \Lambda(x) \circ \Lambda(y)$$
$$\Lambda([c]) = [c] \otimes 1$$

More specifically, for an element $z \in H_k(X)$ with coproduct $\Delta z = \sum z' \otimes z''$, we have the identities

$$\Lambda(x + y)(z) = \sum (\Lambda x)(z') \# (\Lambda y)(z''),$$
$$\Lambda(xy)(z) = \sum (\Lambda x)(z') \circ (\Lambda y)(z'').$$

For $z \in H_k(X)$ with augmentation $\epsilon(z) \in H_k(\ast)$ and $c \in E^n$, we have

$$\Lambda([c])(z) = \epsilon(z)[c].$$

**Proof.** Given elements $x, y \in E^n(X)$, represented by maps $X \to E_n$, the sum is represented by the composite

$$X \xrightarrow{\Lambda} X \times X \xrightarrow{(x,y)} E_n \times E_n \xrightarrow{\#} E_n.$$ 

Similarly, a product is represented by a composite

$$X \xrightarrow{\Lambda} X \times X \xrightarrow{(x,y)} E_p \times E_q \xrightarrow{\circ} E_{p+q}.$$
and a constant \( c \in E^n \) by a composite

\[
X \to \ast \xrightarrow{\ c \ } E_n.
\]

The desired identities follow by applying \( H_* \). \( \square \)

**Remark 3.2.4.** In particular, for \( X = \mathbb{CP}^\infty \) with mod-\( p \) graded cohomology ring \( \mathbb{F}_p[t] \), we can view the unstable power operation as a map

\[
E^n(\mathbb{CP}^\infty) \to H_*(E_n)\mathbb{F}[t].
\]

When \( E \) is complex oriented, the orientation class \( x \in E^2(\mathbb{CP}^\infty) \) is taken to the power series

\[
\Lambda(x) = \sum_{i \geq 0} b_i t^i \in H_*(E_2)\mathbb{F}[t]
\]

(Definition 3.1.1) denoted by \( b(t) \) in [RW74]. In these terms, Ravenel–Wilson’s identity

\[
b(s + t) = b(s) + \# [a_{i,j}] \circ b(s)^{a_i} \circ b(t)^{a_j}
\]

is proved by first applying \( \Lambda \) to the identity \( m^*(t) = \sum a_{i,j} t^j \) in \( E^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \) and then using naturality of \( \Lambda \) to write \( \Lambda m^*(t) = m^* b(t) = b(s + t) \).

While we will not require it, it can be clarifying to examine a “reduced” version of this invariant, especially in cases where \( X \) has a basepoint. We begin by observing that \( \Lambda(\alpha) - [0] \) takes values in reduced homology for any \( \alpha \in E^*(X) \).

**Definition 3.2.5.** Let \( E \) be a multiplicative generalized cohomology theory represented by an \( \Omega \)-spectrum \( \{E_n\}_{n \in \mathbb{Z}} \). The **reduced unstable homology invariant** for \( E \)-cohomology is the natural transformation of sets

\[
\lambda: E^n(X) = [X, E_n] \to \text{Hom}(H_* X, H_* E_n) \cong \widetilde{H}_*(E_n) \otimes H^*(X)
\]

given by \( \lambda(\alpha) = \Lambda(\alpha) - [0] \).

The identities for the operator \( \Lambda \) translate into ones for \( \lambda \) which are particularly transparent when taken mod decomposables for \( \# \).

**Proposition 3.2.6.** The reduced unstable homology invariant \( \lambda \) satisfies the following formulas:

\[
\begin{align*}
\lambda(x + y) &= \lambda(x) + \lambda(y) + \lambda(x) \# \lambda(y) \\
\lambda(xy) &= \lambda(x) \circ \lambda(y) \\
\lambda([c]) &= [c] - [0]
\end{align*}
\]

The composite map

\[
E^*(X) \xrightarrow{\lambda} \widetilde{H}_*(E_*) \otimes H^*(X) \to (Q^* \widetilde{H}_*(E_*)) \otimes H^*(X)
\]

which reduces mod \#-decomposables is a natural homomorphism of graded \( E^* \)-algebras.
Finally, we consider the case of reduced cohomology.

**Proposition 3.2.7.** Suppose \( \alpha \in \tilde{E}^n(X) \) corresponds to a based map \( X \to E_n \). Then the reduced unstable invariant \( \lambda(\alpha) \) naturally takes values in \( \tilde{H}_*E_n \otimes \tilde{H}^*(X) \).

**Proof.** There is a restriction map \( \tilde{H}_*E_n \otimes \tilde{H}^*(X) \to \tilde{H}_*E_n \otimes F_p \) induced by the inclusion of the basepoint \( * \to X \). An element \( \alpha \in E^n(X) \) which restricts to an element \( c \in E^n(*) \) at the basepoint is sent to the element \( \lambda(\alpha) = \Lambda(\alpha) - [0] \) which restricts to \( [c] - [0] \in \tilde{H}_*E_n \). If the map is based, then \( c = 0 \) and so \( \lambda(\alpha) \) lifts to the tensor with reduced cohomology. \( \square \)

### 3.3 Unit groups

For a ring spectrum \( E \), the space \( SL_1(E) \subset \Omega^\infty E \) of strict units is the path component of the multiplicative unit \( 1 \in \pi_0(E) \). This construction is functorial in \( E \). If we define \( \tilde{E}_0 \subset E_0 \) to be the path component of 0, then there is a homotopy equivalence \( \tilde{E}_0 \to SL_1(E) \) given by applying \([1]#(-)\). In particular, there are canonical isomorphisms \( \pi_k(SL_1(E)) \cong \pi_k(E) \) for \( k > 0 \) and \( H_k(SL_1(E)) \cong H_k(\tilde{E}_0) \). When \( E \) is an \( E_\infty \)-algebra, the space of units inherits a corresponding structure.

**Theorem 3.3.1** ([May77, IV.1.8]). For \( E \) an \( E_\infty \)-algebra, the space \( SL_1(E) \) has a natural structure of an infinite loop space such that the map

\[
\Sigma_\infty^\ast SL_1(E) \to E
\]

is a natural map of \( E_\infty \)-algebras.

**Proposition 3.3.2.** Suppose \( E \) is an \( E_\infty \)-algebra, \( HR \) is an Eilenberg-Mac Lane spectrum for a commutative ring \( R \), and \( E \to HR \) is a map of \( E_\infty \)-algebras. Then there is a natural suspension map

\[
\sigma : SL_1(E) \to \Omega SL_1(HR \wedge HR)_E
\]

of infinite loop spaces realizing, for \( k > 0 \), the natural map \( \pi_k(E \to \text{Tor}^E_{1,k}(R,R) \) in the Küneth spectral sequence

\[
\text{Tor}^E_{1,k}(R,R) \Rightarrow \pi_* \left( HR \wedge_E HR \right)
\]

of [EKMM97, IV.4.1].

**Proof.** Since \( SL_1 \) only depends on connective covers, without loss of generality we can assume that \( E \) is connective. We consider the commutative diagram

\[
\begin{array}{ccc}
E & \to & HR \\
\downarrow & & \downarrow \\
HR & \to & HR \wedge_E HR.
\end{array}
\]

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We then apply $SL_1$ to this diagram. The space $SL_1(HR)$ is contractible, so the commutative diagram of infinite loop spaces

$$
\begin{array}{ccc}
SL_1E & \longrightarrow & SL_1(HR) \\
\downarrow & & \downarrow \\
SL_1(HR) & \longrightarrow & SL_1(HR \wedge E HR)
\end{array}
$$

determines (up to contractible indeterminacy) two nullhomotopies of the diagonal map as infinite loop space maps. Gluing these nullhomotopies together gives a map of infinite loop spaces

$$SL_1(E) \rightarrow \Omega SL_1(E) \rightarrow \Omega SL_1(HR \wedge E HR).$$

To show compatibility with the Künneth spectral sequence, we begin by recalling its construction. Setting $HR = M_0$, we iteratively fiber sequences

$$M_{i+1} \rightarrow F_i \rightarrow M_i$$

of $E$-modules which are exact on homotopy groups, where $F_i \simeq \vee \alpha \Sigma^n E$ is a free graded $E$-module, and smash over $E$ with $HR$; the resulting long exact sequences assemble into an exact couple that calculates $\pi_*(HR \wedge E HR)$ with $E_2$-term the desired Tor-groups. In particular, we may choose the unit map $E \rightarrow HR$ as one of the factors in the map $F_0 \rightarrow M_0$, which gives us a map $\Sigma^\infty SL_1(E) \rightarrow E \rightarrow F_0$.

Let $\beta : S^k \rightarrow SL_1(E)$ represent an element in $\pi_k SL_1(E)$ for $k > 0$, and consider the diagram

$$
\begin{array}{ccc}
\Sigma^\infty SL_1(E) & \longrightarrow & \Sigma^\infty SL_1(E) \\
\downarrow & & \downarrow \\
M_1 & \longrightarrow & HR
\end{array}
$$

whose rows are fiber sequences and where the dotted arrow is the map induced by the map $\sigma$. The composite map $S^k \rightarrow \Sigma^\infty SL_1(E) \rightarrow M_1$ represents the element $\beta \in \ker(\pi_k E \rightarrow \pi_k HR)$, and lifts to a map $S^k \rightarrow F_1$. The image in $\pi_k(HR \wedge E F_1)$ is the element corresponding to $\beta$ in $\text{Tor}_{1,k}^E(R, R)$. However, this also coincides with the suspension of $\beta$ under the dotted arrow that uses the two nullhomotopies of $\Sigma^\infty SL_1(E) \rightarrow HR$.

\begin{corollary}
For a ring $R$, there are suspension maps

$$\sigma : \widetilde{H}_*(SL_1(E); R) \rightarrow H_*(BSL_1(E); R) \rightarrow \pi_* \left( HR \wedge E HR \right).$$

These are natural in maps $E \rightarrow HR$ of $E_\infty$-algebras, and on the Hurewicz image of $\pi_*(BSL_1(E))$ these are given by the suspension map. When $R = F_2$, this map commutes with the Dyer–Lashof operations.
\end{corollary}
Proof. The map $SL_1(E) \to \Omega SL_1(HR \wedge HR)$ is adjoint to a map $BSL_1(E) \to SL_1(HR \wedge HR)$ of infinite loop spaces. We begin with the map of $E_\infty$-algebras

$$\Sigma^\infty_+ BSL_1(E) \to \Sigma^\infty_+ SL_1(HR \wedge HR) \to HR \wedge HR.$$ 

The adjunction between $E_\infty$-algebras and $E_\infty HR$-algebras (using the left unit $HR \to HR \wedge E HR$) then produces a natural map

$$HR \wedge BSL_1(E)_+ \to HR \wedge HR$$

of $E_\infty HR$-algebras realizing our desired map. In particular, if $R = \mathbb{F}_2$ this map of $H$-algebras commutes with the Dyer–Lashof operations.

Corollary 3.3.4. In the commutative diagram

$$H_*(SL_1(MU); Q) \leftarrow H_*(SL_1(MU); Z) \to H_*(SL_1(MU))$$

$$\downarrow \quad \downarrow$$

$$\pi_{*+1}(HZ \wedge_\mu HZ) \leftarrow \pi_{*+1}(HZ \wedge_\mu HZ) \to \pi_{*+1}(H_\mu H),$$

where the vertical maps are suspensions, the left-hand horizontal arrows are injective and the right-hand top horizontal arrow is surjective. In particular, the suspension map in mod-$2$ homology is determined by the rational suspension map. In addition, the right-hand vertical map preserves the Dyer–Lashof operations.

Proof. The injectivity and surjectivity of the top rows was shown in Corollary 3.1.3. The injectivity of the bottom-left map follows because the comparison map of Künneth spectral sequences

$$Tor^\pi_{*+1}(\mathbb{Z}, \mathbb{Z}) \to Tor^\pi_{*+1}(Q, Q)$$

becomes an inclusion of exterior algebras $\Lambda[\sigma x_i] \to Q \otimes \Lambda[\sigma x_i]$, and both spectral sequences degenerate at the $E_2$-term. Therefore, the map $\pi_{*+1}(HZ \wedge_\mu HZ) \to \pi_{*+1}(H_\mu H)$ is injective.

We can now examine the properties of the rational suspension map by using the rational Hopf ring.

Proposition 3.3.5. In the rational Hopf ring, the suspension map

$$\tilde{H}_*(SL_1(MU); Q) \to H_{*+1}(HZ \wedge_\mu HZ),$$

in terms of the Ravenel–Wilson basis, is a composite

$$\tilde{H}_*(SL_1(MU); Q) \to Q^*Q^* \tilde{H}_*(MU_0; Q)/(b_2, b_3, \ldots) \to \pi_{*+1}(HZ \wedge_\mu HZ)$$

that kills $\#$-decomposables, $\circ$-decomposables, and $b_i$ for $i > 1$, and sends any of the remaining basis elements $[\alpha] \circ b_i^{\#}$ to the suspension class $\sigma \alpha$ in the Künneth spectral sequence from Proposition 2.6.5A.
Proof. There is a commutative diagram of $E_{\infty}$ rings over $HQ$:

$$\Sigma^\infty_{+} SL_1(MU) \rightarrow HQ \wedge SL_1(MU)_{+}$$

$$MU \rightarrow HQ \wedge MU$$

Applying the natural map $\tilde{H}_*(SL_1(-); \mathbb{Q}) \rightarrow HQ \wedge (-) HQ$, we find that the suspension map

$$\tilde{H}_*(SL_1(MU); \mathbb{Q}) \rightarrow \pi_{*+1}(HQ \wedge HQ) \equiv \pi_{*+1}(HQ \wedge HQ)$$

can be computed as the composite

$$\tilde{H}_*(SL_1(MU); \mathbb{Q}) \rightarrow H_*(MU; \mathbb{Q}) \rightarrow H_{*+1}(HQ \wedge HQ).$$

The first map, under the isomorphism $[-1] \# (-): \tilde{H}_*(SL_1(MU)) \equiv \tilde{H}_*(MU)$, sends $\#$-decomposable elements to zero, carries $\circ$-products to products, and takes the elements $b_i$ for $i > 1$ to $\circ$-decomposable elements $b_i = [a_i] \circ b_i^{\circ}$ due to the Ravenel–Wilson relation (3.1). The second is the suspension map $\sigma$, which carries $\circ$-decomposable elements to zero. The element $[\alpha] \circ b_i^{\circ}$ is the Hurewicz image of $\alpha$ which, by definition, is carried to the suspension class $\sigma \alpha$.

Taking this together with Corollary 3.3.4, we find the following.

**Corollary 3.3.6.** The suspension map

$$\tilde{H}_*(SL_1(MU)) \rightarrow \pi_{*+1}\left(H \wedge H_{MU}\right)$$

on mod-$2$ homology, in terms of the Ravenel–Wilson basis, is a composite

$$\tilde{H}_*(SL_1(MU)) \rightarrow Q^*Q^*H_*(MU)/\langle b_2, b_3, \ldots \rangle \rightarrow \pi_{*+1}\left(H \wedge H_{MU}\right)$$

that kills $\#$-decomposable, $\circ$-decomposable, and $b_i$ for $i > 1$, and sends any of the remaining elements $[\alpha] \circ b_i^{\circ}$ in the Ravenel–Wilson basis to the suspension class $\sigma \alpha$ from the Künneth spectral sequence.

**Proposition 3.3.7.** The suspension map $\sigma$ on mod-$2$ homology commutes with Dyer–Lashof operations.

Proof. This map is the composite

$$\tilde{H}_*(SL_1(MU)) \rightarrow H_{*+1}(BSL_1(MU)) \rightarrow \pi_{*+1}\left(H \wedge H_{MU}\right).$$

The Dyer–Lashof operations on the homology of infinite loop spaces are stable, and hence preserved by the first map; the compatibility of the second map is Corollary 3.3.3.

□
4 Power operations

4.1 Power operations in complex oriented theories

In this section we will recall the work from [BMMS86] on power operations in cohomology theories, and specifically results on $H_\infty$-algebra structures from [BMMS86, VIII].

For an $E_\infty$ (and hence $H_\infty$) ring spectrum $E$, the $E$-cohomology of a (based) space $X$ has natural power operations as follows. Fix $m > 0$ and write $D_m$ for the extended power functor given by

$$D_m(Y) = (Y^m)^{hE_m}.$$ 

Representing an element $\alpha \in E^0(X)$ as a map $\alpha: \Sigma^\infty X \to E$, we form the commutative diagram

$$\begin{array}{ccc}
\Sigma^\infty D_m X & \xrightarrow{D_m \alpha} & D_mE \\
\downarrow & & \downarrow \\
X \wedge (B\Sigma_m)_+ & \xrightarrow{P_m(\alpha)} & E,
\end{array}$$

where the right-hand map is induced by the multiplicative structure on $E$. In particular, this produces natural power operations:

$$P_m: \widetilde{E}^0(X) \to \widetilde{E}^0(D_mE)$$

$$P_m: \widetilde{E}^0(X) \to \widetilde{E}^0(X \wedge (B\Sigma_m)_+)$$

These are multiplicative in an appropriate sense, and by replacing $X$ with $X_+$ we obtain compatible unbased versions:

$$P_m: E^0(X) \to E^0((X^m)^{hE_m})$$

$$P_m: E^0(X) \to E^0(X \times B\Sigma_m)$$

Outside degree 0, we cannot draw conclusions which are as strong in general. Given an element $\alpha \in \widetilde{E}^n(X)$ represented by a map $\Sigma^n X \to E \wedge S^n$, we can only define part of the desired diagram:

$$\begin{array}{ccc}
\Sigma^n D_m X & \xrightarrow{D_m \alpha} & D_m(E \wedge S^n) \\
\downarrow & & \downarrow \\
\Sigma^n X \wedge (B\Sigma_m)_+ & \xrightarrow{P_m(\alpha)} & E \wedge S^{nm}
\end{array}$$

With extra structure on $E$ we can complete this diagram when $n$ is a multiple of some fixed constant $d$: this is the case where $E$ is $H^d_\infty$-algebra [BMMS86, I.4]. An $H^d_\infty$-algebra is an algebra equipped with explicit extra structure maps $D_m(E \wedge S^n) \to E \wedge S^{dn}$, multiplicative and compatible across $n$ and $m$. These allow us to obtain
power operations:

\[
\mathcal{P}_m: \tilde{E}^{dk}(X) \to \tilde{E}^{dmk}(D_m X)
\]

\[
P_m: \tilde{E}^{dk}(X) \to \tilde{E}^{dmk}(X \wedge (B\Sigma_m)_+)
\]

These are multiplicative, and replacing \(X\) with \(X_+\) gives compatible unbased versions:

\[
\mathcal{P}_m: E^{dk}(X) \to E^{dmk}((X^m)_{h\Sigma_m})
\]

\[
P_m: E^{dk}(X) \to E^{dmk}(X \times B\Sigma_m).
\]

Cohomology is representable, so we may applying the Yoneda lemma. Restricting to the case where \(m\) is a chosen prime \(p\) and \(d = 2\), we get the following.

**Theorem 4.1.1.** If \(E\) is an \(H_\infty\)-algebra, there are natural based and unbased power operations for \(n \in \mathbb{Z}\):

\[
P: \tilde{E}^{2n}(X) \to \tilde{E}^{2pn}(X \wedge (B\Sigma_p)_+)
\]

\[
P: E^{2n}(X) \to E^{2pn}(X \times B\Sigma_p)
\]

These are universally represented by maps of based spaces \(E_{2n} \wedge (B\Sigma_p)_+ \to E_{2pn}\), and satisfy \(P(x)P(y) = P(xy)\).

For instance, the complex bordism spectrum \(MU\) is an \(H_\infty\)-algebra [BMMS86, VIII.5.1], giving us power operations on even-degree classes previously studied by tom Dieck and Quillen [tD68, Qui71] that extend the power operations in degree zero. The spectrum \(MU\), which is complex oriented and has canonical Thom classes for complex vector bundles, also has the special property that these operations are compatible with the Thom isomorphism, as described by Quillen.

**Proposition 4.1.2** ([Qui71]). For any complex vector bundle \(\xi \to X\) of dimension \(k\), write \(t(\xi) \in MU^{2k}(Th(\xi))\) for the canonical Thom class of \(\xi\) and \(e(\xi) \in MU^{2k}(X_+)\) for the Euler class.

The based operation \(\mathcal{P}_m\) preserves Thom classes: we have

\[\mathcal{P}_m(t(\xi)) = t(D_m \xi),\]

where \(D_m \xi\) is the extended power bundle over \((X^m)_{h\Sigma_m}\). Restricting along the diagonal, we have

\[P_m t(\xi) = e(\xi \otimes \overline{\rho}) t(\xi)\]

where \(\overline{\rho}\) is the bundle on \(B\Sigma_m\) induced by the reduced permutation representation of \(\Sigma_m\) and \(\xi \otimes \overline{\rho}\) is the exterior tensor bundle on \(X \times B\Sigma_m\). In particular, the Thom isomorphism fits into a commutative diagram

\[
\begin{array}{ccc}
MU^{2n}(X) & \xrightarrow{P_m} & MU^{2mn}(X \times B\Sigma_m) \\
\downarrow t(\xi) & & \downarrow e(\xi \otimes \overline{\rho}) t(\xi) \\
MU^{2(n+k)}(Th(\xi)) & \xrightarrow{P_m} & MU^{2m(n+k)}(Th(\xi) \wedge (B\Sigma_m)_+) \\
\end{array}
\]
The cohomology of symmetric groups is closely related to formal group law theory [Qui71], and in particular the effect of the power operation $P$ on the canonical first Chern class $x \in \tilde{MU}^2(\mathbb{CP}^\infty)$ was determined by Ando [And95].

**Theorem 4.1.3.** The inclusion $C_p \hookrightarrow \Sigma_p$ induces inclusions:

\[
\begin{align*}
\text{MU}^*(B\Sigma_p) &\hookrightarrow \text{MU}^*[\alpha]/[p]_F(\alpha) \\
\text{MU}^*(\mathbb{CP}^\infty \times B\Sigma_p) &\hookrightarrow \text{MU}^*[x,\alpha]/[p]_F(\alpha)
\end{align*}
\]

In these coordinates, the power operation $P$ satisfies $P(x) = \prod_{i=0}^{p-1} (x + [i]_F(\alpha))$.

The power operations $P: \text{MU}^{2k} \rightarrow \text{MU}^{2pk}(B\Sigma_p)$ are in principle determined by these results, naturality, and multiplicativity, and are closely related to the Lubin isogeny in the theory of formal group laws. However, it has been difficult to obtain closed-form expressions for these power operations. We will require the following computation of Johnson–Noel, using the fact that the generator $x_2$ of the complex cobordism ring in dimension 4 is $\mathbb{CP}^2$.

**Theorem 4.1.4 ([JN10, 6.3]).** The polynomial generator $x_2$ of the complex bordism ring $\text{MU}_* \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$, appearing in $\pi_4(\text{MU})$, has image

\[
P(x_2) = \alpha^2(v_1^0 + v_2^0) + \alpha^3(v_1^1 + v_3)
\]

in $BP_*[[\alpha]]/([2]_F(\alpha), \alpha^8)$, where $P$ is the $2$-primary power operation. In particular, $P(x_2) \equiv x_7\alpha^7 \mod$ decomposables and higher order terms in $\alpha$.

**Remark 4.1.5.** The powers of $\alpha$ appearing in the above result differ by a shift from those in [JN10] due to a change in how gradings are applied.

The main result of this paper hinges on this theorem. In Appendix A we will show that Johnson–Noel’s method can be adapted to one that works in torsion-free quotients of the Lazard ring. This tweak allows us to give an abbreviated version of their proof at the prime 2, ignoring decomposables, that is easier to carry out without computer assistance.

### 4.2 Unstable Dyer–Lashof operations

We recall the computation of the cohomology of the symmetric group $\Sigma_p$:

\[
H^*(B\Sigma_p) \cong \begin{cases} 
\mathbb{F}_2[u] & \text{if } p = 2, \\
\mathbb{F}_2[u] \otimes \Lambda[v] & \text{if } p > 2.
\end{cases}
\]

Here $u$ has degree 1 if $p = 2$, while $u$ has degree $2p - 2$ and $v$ has degree $2p - 3$ if $p$ is odd.

**Definition 4.2.1.** If $E$ is an $H^\infty$-algebra, the homology power operation

\[
Q: H_*(E_{2m}) \rightarrow H_*(E_{2pn}) \otimes H^*(B\Sigma_p)
\]
is adjoint to the map

\[ H_*P : H_*(E_{2n}) \otimes H_*(B \Sigma_p) \to H_*(E_{2pn}) \]

induced by the map \( E_{2n} \wedge (B \Sigma_p)_+ \to E_{2pn} \) of based spaces from Theorem 4.1.1.

The multiplicity of the natural power operation \( P \) has the following consequence.

**Proposition 4.2.2.** The operation \( Q \) satisfies \( Q(x) \circ Q(y) = Q(x \circ y) \) and \( Q([0]) = [0] \).

**Proposition 4.2.3.** Suppose \( E \) is an \( H^2 \)-algebra. Then for all \( n \in \mathbb{Z} \) we have a commutative diagram of sets

\[
\begin{array}{ccc}
\tilde{E}^{2n}(X) & \xrightarrow{P} & \tilde{E}^{2pn}(X \wedge (B \Sigma_p)_+) \\
\wedge & \downarrow & \wedge \\
H_*(E_{2n}) \wedge H^*(X) & \xrightarrow{Q \otimes 1} & H_*(E_{2pn}) \wedge H^*(B \Sigma_p) \wedge H^*(X)
\end{array}
\]

that is natural in \( X \). The horizontal maps preserve products and the bottom map is a map of abelian groups.

**Proof.** The power operation \( P \) sends an element represented by a map \( \alpha : X \to E_{2n} \) to the composite

\[ P(\alpha) : X \wedge (B \Sigma_p)_+ \xrightarrow{\alpha \wedge 1} E_{2n} \wedge (B \Sigma_p)_+ \xrightarrow{P} E_{2pn}. \]

The value of \( \Lambda(P(\alpha)) \) is the effect on homology, which is the composite

\[ H_*(X) \otimes H_*(B \Sigma_p) \xrightarrow{H_*\alpha \otimes 1} H_*(E_{2n}) \otimes H_*(B \Sigma_p) \xrightarrow{H_*P} H_*(E_{2pn}). \]

Taking adjoints recovers the statement about completed tensor products. \( \square \)

**Remark 4.2.4.** The map \( BC_p \to \mathbb{CP}^\infty \) induces a map \( \tilde{E}^2(\mathbb{CP}^\infty) \to \tilde{E}^2(BC_p) \) that takes the orientation class \( x \) to the generator \( \alpha \) described in Theorem 4.1.3, and the map \( H^*(\mathbb{CP}^\infty) \to H^*(BC_p) \) is the map \( \mathbb{F}_p[t] \to \mathbb{F}_p[u] \) sending \( t \) to \( u \) if \( p \) is odd and \( u^2 \) if \( p = 2 \). By naturality of \( \Lambda \), we find that the element \( \Lambda(\alpha) \) is equal the image \( b(u^2) \) of \( b(t) \) if \( p = 2 \).

For the remainder of this section we will focus on the prime 2. We first recall the following calculation, which is dual to the identity used to define the Steenrod operations in [Ste62, VII.3.2, VII.6.1].

**Lemma 4.2.5.** For a space \( X \) with second extended power \( D_2(X) \), the composite diagonal map

\[ H_*(X) \otimes H_*(B \Sigma_2) \to H_*(X \times B \Sigma_2) \to H_*(D_2(X)) \]

on mod-2 homology is given by

\[ v \otimes \beta_n \mapsto \sum_{j \geq 0} Q^{j+n}(P_j v). \]

Here \( \beta_n \) is dual to \( u^n \) and \( P_j \) is the homology operation dual to \( Sq^j \).
As a result, the Dyer–Lashof operations can be recovered from this diagonal map into the extended power.

**Theorem 4.2.6.** Consider the homology operations
\[ Q: H_*(MU_{2n}) \to H_*(MU_{4n}) \otimes H^*(B\Sigma_2) \]
from Definition 4.2.1. Then there are multiplicative Dyer–Lashof operations
\[ \hat{Q}^s: H_*(MU_{2n}) \to H_*(MU_{4n}) \otimes H^*(B\Sigma_2), \]
extending the Dyer–Lashof operations in degree zero of [CLM76, II.1] (coming from the multiplicative \( E_\infty \)-space structure) to Dyer–Lashof operations in even degrees. These satisfy the Cartan formula
\[ \hat{Q}^s(x \circ y) = \sum_{p+q=s} \hat{Q}^p(x) \circ \hat{Q}^q(y) \]
and are related to \( Q \) by the identity
\[ Q(x) = \sum_{n,j} \hat{Q}^{j+n}(P_j x)u^n. \]

In particular, if all nontrivial Steenrod operations vanish on \( x \) then \( Q(x) = \sum \hat{Q}^n(x)u^n \).

This property is invariant under the product \( \circ \).

### 4.3 Power operations in the Hopf ring

We can now begin to use the results of the previous sections to calculate multiplicative Dyer–Lashof operations in the Hopf ring for \( MU \) (the additive ones having been determined by Turner [Tur93]). First we will find the effect on the class \( b_1 \in H_2(MU_2) \) of Definition 3.1.1, because \( \circ \)-multiplication by \( b_1 \) represents suspension.

**Proposition 4.3.1** (cf. [Pri75]). Let \( b_k \in H_{2k}(MU_2) \) denote the fundamental classes of Definition 3.1.1. Then the 2-primary multiplicative Dyer–Lashof operations satisfy
\[ \hat{Q}^{2n}b_1 = b_1 \circ b_n \]
for all \( n \geq 1 \).

**Proof.** For a general prime \( p \), we consider the commutative diagram
\[
\begin{array}{ccc}
MU^0(BU(1)) & \xrightarrow{P} & MU^0(BU(1) \times B\Sigma_p) \\
\downarrow t(y_1) & & \downarrow e(y_1 \otimes \overline{\iota}(y_1)) \\
\overline{MU}^2(MU(1)) & \xrightarrow{P} & MU^{2p}(MU(1) \wedge (B\Sigma_p)_+) \\
\downarrow \Lambda & & \downarrow \Lambda \\
H_*(MU_2) \otimes H^*(MU(1)) & \xrightarrow{Q \otimes 1} & H_*(MU_{2p}) \otimes H^*(B\Sigma_p) \otimes H^*(MU(1)),
\end{array}
\]
where the top square expressing compatibility of $P$ with the Thom isomorphism is from Proposition 4.1.2. Because $x$ is the Thom class of the canonical bundle on $BU(1)$, $\Lambda(t(y_1)) = b(s)$. The image of the unit $1 \in MU^0(BU(1))$ along the left-to-bottom composite is then

$$(Q \otimes 1)(\Lambda(x)) = (Q \otimes 1)(b(s)) = \sum Q(b_k)s^k.$$  

On the other hand, the image along the top-right composite is

$$\Lambda \left( x \prod_{k=1}^{p-1} (x + [k][F]_\alpha) \right) = b(s) \circ (b(s) + [F] b(u^2)) \circ \cdots \circ (b(s) + [F] [p-1][F] b(u^2)),$$

using the expression for the Euler class of the exterior tensor bundle $\gamma_1 \otimes \mathbb{P}$ on $BU(1) \times \Sigma^\infty\Sigma^\infty$.  

Taking the coefficient of $s$, which involves only the linear term of $b(s)$ and the constant coefficients (in terms of $s$) of the factors $b(s) + [F] [k][F] b(u^2)$, we find that

$$Q(b_1) = b_1 \circ b(u^2) \circ [2][F] b(u^2) \circ \cdots \circ [p-1][F] b(u^2).$$

When we specialize to $p = 2$ and apply Theorem 4.2.6, we find

$$\sum_{j \geq 2} \mathcal{Q}^j(b_1)u^j = b_1 \circ b(u^2) = \sum_{n \geq 1} (b_1 \circ b_n)u^{2n}$$

as desired.  

\textbf{Proposition 4.3.2.} Suppose that $y \in \pi_{2n} MU$ and that, in the coordinates of Theorem 4.1.3, we have

$$P(y) = \sum_{i=0}^{\infty} c_i a^i$$

for some elements $c_i \in \pi_{4n+2i} MU$. Then

$$Q([y]) = \# \left[ c_i \right] \circ b(u^2)^{n^i}$$

\textbf{Proof.} Taking $X = *$ in Proposition 4.2.3 identifying $[y]$ with $\Lambda(y)$, we find

$$Q([y]) = \Lambda(P(y)) = \Lambda \left( \sum_{i=0}^{\infty} c_i a^i \right) = \# \left[ c_i \right] \circ b(u^2)^{n^i}$$

by Proposition 3.2.3 and Remark 4.2.4.  

\textbf{Corollary 4.3.3.} Mod $\#$-decomposables and the $\circ$-ideal generated by $b_2, b_3, \ldots$, the Hurewicz image $[x] \circ b_i^{c_n} \in H_{2n}(MU_0)$ of $x \in \pi_{2n}(MU)$ satisfies

$$Q([x] \circ b_i^{c_n}) \equiv \sum_{i=0}^{\infty} [c_i] \circ (b_i)^{c(i+2n)} u^{2(i+n)}.$$

In particular, $\mathcal{Q}^{2k}([x] \circ b_i^{c_n}) \equiv [c_{k-n}] \circ b_i^{c(k+n)}$ in this quotient.
Proof. The first part follows from the multiplication formula $Q([x] \circ b_1^n) = Q([x]) \circ Q(b_1)^n$. The second part follows from Theorem 4.2.6 and the fact that the operations $P_j$ vanish on $[x] \circ (b_1)^n$ for $j > 0$. □

4.4 Power operations in the $MU$-dual Steenrod algebra

We will now apply the previous technology to compute a multiplicative Dyer–Lashof operation in $H^*(SL_1(MU))$. In order to do so, we need some preliminary results about how the additive product interacts with multiplicative Dyer–Lashof operations.

**Proposition 4.4.1.** At $p = 2$, the multiplicative and additive Dyer–Lashof operations in the Hopf ring of an $E_\infty$-algebra satisfy the following identities.

1. When $x$ and $y$ are in the homology of the path component of zero, we have
   \[ \widehat{Q}^s(x \# y) \equiv Q^s(x \circ y) \mod \#\text{-decomposables}. \]

2. When $y$ is in the homology of the path component of zero, we have
   \[ \widehat{Q}^s([1] \# y) \equiv [1] \# Q^s(y) + [1] \# \widehat{Q}^s(y) \mod \#\text{-decomposables}. \]

3. For any $x$ there exist elements $z_i$ for $0 < i < |x|$ such that
   \[ Q^s(x) = Q^s[1] \circ x + \sum Q^{s+i}[1] \circ z_i. \]
   In particular, $Q^s(x)$ is $\circ$-decomposable for any $x$ and any $s > 0$.

Proof. For the first item, the mixed Cartan formula [CLM76, II.2.5] takes the form
\[
\widehat{Q}^s(x \# y) = \sum_{p+q+r=s} \sum \widehat{Q}^p(x') \# Q^q(x'' \circ y') \# \widehat{Q}^r(y'').
\]
In particular, the only time this is not decomposable under $\#$ is when both $Q^p(x')$ and $Q^r(y'')$ are of degree zero; this occurs when $p = r = 0$ and we take the terms $[0] \otimes x$ and $y \otimes [0]$ of the coproduct.

Similarly, for the second item the only nonzero terms in the mixed Cartan formula occur when $p = 0$ and either $y' = [0]$ or $y'' = [0]$.

The third item is proved by induction on the degree of $x$, using the formula
\[
Q^s(1) \circ x = \sum Q^{s+i}(1 \circ P_i x)
\]
from [CLM76, II.1.6]. □

The above shows that essentially all the cross terms arising in the determination of $\widehat{Q}([1] \# x)$ become decomposable under $\circ$. (This closely parallels the proof that the power operation $P$ is additive mod the ideal generated by the transfer of 1.)
Proposition 4.4.2. Suppose that \( y \in \pi_{2n}MU \) and that, in the coordinates of Theorem 4.1.3, we have

\[
P(y) = \sum_{i=0}^{\infty} c_i x^i
\]

for some elements \( c_i \in \pi_{4n+2i}MU \). Then \( \text{mod } \# \)-decomposables, \( \circ \)-decomposables, and the ideal generated by \( b_2, b_3, \ldots \), the Hurewicz image \([1] \# ([x] \circ b_1^n)\) of \( x \in \pi_{2n}SL_1 MU \) satisfies

\[
Q([1] \# ([x] \circ b_1^n)) \equiv \sum_{i=0}^{\infty} [c_i] \circ (b_i)^{(i+2n)} u^{2(i+n)}.
\]

In particular, \( \hat{Q}^{10}([1] \# ([x] \circ b_1^n)) \equiv [c_{k-n} \circ b_1^{n(k+n)}] \) in this quotient.

We can now apply the results of Johnson–Noel from Theorem 4.1.4, as well as Corollary 3.3.6 and Proposition 3.3.7.

Corollary 4.4.3. The Dyer–Lashof operations in \( H, SL_1(MU) \) satisfy

\[
\hat{Q}^{10}([1] \# ([x_2] \circ b_1^{n_2})) \equiv [1] \# ([x_7] \circ b_1^{n_7})
\]

mod \( \# \)-decomposables, \( \circ \)-decomposables, and the ideal \( (b_2, b_3, \ldots) \).

The Dyer–Lashof operations in \( \pi_*(H \wedge_{MU} H) \) satisfy

\[
Q^{10}(\sigma x_2) = \sigma x_7.
\]

Remark 4.4.4. We can take a brief pause to explain why this should show that no map \( MU \to BP \) can be given the structure of a map of \( E_1 \)-algebras at the prime \( 2 \), extending [JN10]. If it could, then we should be able to obtain a map of \( E_0 \)-algebras \( H \wedge_{MU} H \to H \wedge_{BP} H \), on homotopy given by a map of exterior algebras \( \Lambda[\sigma x_1] \to \Lambda[\sigma x_2, \ldots] \). However, this map would be zero on the element \( \sigma x_2 \) and nonzero on the element \( \sigma x_7 = Q^{10}(\sigma x_2) \). (Here we use that \( Q^{10} \), on a class in degree \( 5 \), is realized by an operation for \( E_6 \)-algebras—see Remark 2.5.14.)

5 Calculations with \( MU, H\mathbb{Z}/2, \) and \( BP \)

In order to begin with more specific computations of secondary operations, we will use the following convenient definitions.

Definition 5.0.1. For a symbol \( a \) and an integer \( k \), we define \( \mathbb{P}_{H}^{E_n}(a_k) \) to be the free \( E_n \)-algebra \( \mathbb{P}_{H}^{E_n}(S^k) \), writing \( a_k \in \pi_k \mathbb{P}_{H}^{E_n}(a_k) \) for the generator represented by the unit map \( S^k \to \mathbb{P}_{H}^{E_n}(S^k) \).

Similarly, we use the coproduct in \( E_n \)-algebras to define

\[
\mathbb{P}_{H}^{E_n}(a_{k_1}, b_{k_2}, \ldots) = \mathbb{P}_{H}^{E_n}(a_{k_1}) \amalg \mathbb{P}_{H}^{E_n}(b_{k_2}) \amalg \cdots \equiv \mathbb{P}_{H}^{E_n}(\vee S^{k_i})
\]

for a sequence \( (a_{k_1}, b_{k_2}, \ldots) \). If a generator has a known, fixed, degree, we will leave off the subscript.
Definition 5.0.2. Let $D$ be the category of $E_\infty$ $H$-algebras under $\mathcal{P}_H(x)$, where $x$ has degree 2, and $D^n$ the category of $E_n$ $H$-algebras under $\mathcal{P}_H^{E_n}(x)$.

Let $C = D^\pm$ and $C^n = (D^n)^\pm$ as in Definition 2.2.1.

There are forgetful functors between these categories, using the compatible maps $\mathcal{P}_H^{E_n} S^2 \to \mathcal{P}_H^{E_m} S^2$ that are adjoint to the units $S^2 \to \mathcal{P}_H^{E_m} S^2$. The generator of $H_2MU \cong \mathbb{Z}/2$ determines a map $\mathcal{P}_H(x) \to H \wedge MU$ up to equivalence, lifting it to an object of $C$.

5.1 Power operations for $MU$

The 2-primary power operations in $H_\ast MU$ are known by work of Kochman [Koc73], but the following closed-form formula is due to Priddy.

**Theorem 5.1.1 ([Pri75]).** The Dyer–Lashof operations in $H_\ast MU \cong H_\ast BU$ are determined by the following identity:

$$
\sum Q^j b_k = \left(\sum_{n=k}^{\infty} \sum_{u=0}^{k} \binom{n-k+u-1}{u} b_{n+u} b_{k-u} \right) \left(\sum_{n=0}^{\infty} b_n\right)^{-1}
$$

Here $b_0 = 1$ by convention. In particular, we have

$$
\sum Q^1 b_1 = \left(\sum_{n=1}^{\infty} (b_n b_1 + (n-1)b_{n+1}) \right) \left(\sum_{n=0}^{\infty} b_n\right)^{-1}.
$$

This allows the following direct computation. (Compare [Pri75, 2.5], which carries out this computation for $MO$).

**Proposition 5.1.2.** We have the following Dyer–Lashof operations in $H_\ast MU$:

- $Q^2 b_1 = b_1^2$
- $Q^4 b_1 = b_3 + b_1 b_2 + b_1^4$
- $Q^6 b_1 = b_1^6$
- $Q^8 b_1 = b_5 + b_1 b_4 + b_2 b_3 + b_1^2 b_3 + b_1 b_2^2 + b_1^3 b_2 + b_1^5$
- $Q^{10} b_1 = b_7 + b_1^2 b_3 + b_1^2 b_2 + b_1^7$
- $Q^6 b_2 = b_3 + b_1 b_4 + b_2 b_3 + b_1 b_2^2$
- $Q^{10} b_2 = b_7 b_5 + b_1^2 b_4 + b_1^2 b_2 b_3 + b_1^7 b_2^2$

In particular, the following identities hold:

- $0 = Q^6 b_1 + b_1^4$
- $0 = Q^{10} b_1 + (Q^4 b_1)^2$
- $Q^b b_2 = Q^b b_1 + b_1^2 Q^b b_1$
- $0 = Q^{10} b_2 + b_1^2 Q^b b_2$
5.2 Power operations for $H$

The power operations in the dual Steenrod algebra are known by work of Steinberger.

**Theorem 5.2.1** ([BMMS86, III.2.2, III.2.4]). The 2-primary Dyer–Lashof operations in the dual Steenrod algebra satisfy the following identities:

$$1 + \xi_1 + Q^1\xi_1 + Q^2\xi_1 + Q^3\xi_1 + \cdots = (1 + \xi_1 + \xi_2 + \ldots)^{-1}$$

$$Q^s\xi_i = \begin{cases} Q^{s+2^i}\xi_i & \text{if } s \equiv 0, -1 \mod 2^i, \\ 0 & \text{otherwise.} \end{cases}$$

$$Q^s\xi_i = \xi_{i+1}$$

This, again, allowed direct computation.

**Proposition 5.2.2** ([BMMS86, III.5]). We have the following Dyer–Lashof operations in the 2-primary dual Steenrod algebra:

$$Q^2\xi_1 = \xi_2$$
$$Q^3\xi_1 = \xi_3$$
$$Q^4\xi_1 = \xi_4$$
$$Q^5\xi_1 = \xi_5$$
$$Q^{16}\xi_4 = \xi_5$$

In particular, the Cartan formula implies that the following identities hold:

$$0 = Q^6\xi_1 + \xi_1$$
$$0 = Q^8\xi_1 + \xi_1 Q^4\xi_1$$
$$0 = \xi_1 Q^2\xi_1 + (Q^4\xi_1)^2$$

**Remark 5.2.3.** While the identity $Q^{16}\xi_4 = \xi_5$ is valid, the results of this paper only require us to know the easier statement that $Q^{16}\xi_4 \equiv \xi_5 \mod$ decomposable elements.

5.3 Functional operations for $MU \to H\mathbb{Z}/2$

Recall that the category $C$ is the category of $E_\infty$ $H$-algebras under $P_H(x)$, where $x$ has degree 2.

**Theorem 5.3.1.** Consider the maps

$$P_H(x, z_{14}) \xrightarrow{Q} P_H(x, y_{14}) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H$$

in the category $C$, where $Q$ sends $z$ to $Q^{10}y_1 + x^2Q^6y_4$ and $f$ sends $(x, y_{14})$ to $(b_1, b_2)$. Then a functional homotopy operation $(p, f, Q)$ is defined in $P_H(x)$-algebras and satisfies

$$\langle p, f, Q \rangle \equiv \xi_4 \mod$$ decomposables.
Proof. The identities $Q^{10}b_2 + b_2^2Q^6b_2 = 0$ and $p(b_2) = 0$ ensure that there is a homotopy commutative diagram of $E_\infty \mathbb{F}_2(x)$-algebras over $H$:

\[
\begin{array}{ccc}
\mathbb{F}_H(x, z_{14}) & \rightarrow & \mathbb{F}_H(x) \\
\downarrow & & \downarrow \\
\mathbb{F}_H(x, y_4) & \rightarrow & H \wedge MU \\
\downarrow & \downarrow & \downarrow \\
\mathbb{F}_H(x) & \rightarrow & H \wedge H \\
\downarrow & \downarrow & \downarrow \\
& i & \rightarrow H \wedge MU H
\end{array}
\]

In particular, $\overline{Q}$ is a map of augmented objects and $H \wedge MU H$ is a pointed object, ensuring that the secondary operation is defined. As a result, we can define $\langle p, f, \overline{Q} \rangle$ and apply the Peterson–Stein formula of Proposition 2.3.5 to find that there is an identity

\[\langle i, p, f \rangle \overline{Q} = i(p, f, \overline{Q}).\]

(Note that there is no inversion in this Peterson–Stein formula because the target group is a vector space over $\mathbb{F}_2$.)

The bracket $\langle i, p, f \rangle$ takes $y_4$, which maps under $f$ to the Hurewicz image $b_2$ of $x_2 \in \pi_2 MU$, to the suspension class $\sigma b_2$ up to indeterminacy by Proposition 2.6.5. The operation $\overline{Q}$ sends this to $Q^{10}(\sigma b_2) + x^2Q^6(\sigma b_2) = Q^{10}(\sigma b_2)$ because $x$ acts by 0 on $H \wedge MU H$. Then Corollary 4.4.3 implies that $Q^{10}(\sigma b_2) = i(\xi_4)$ mod decomposables, and the map $i$ is an isomorphism on homotopy in degree 15 mod decomposables.

The indeterminacy in the functional homotopy operation $\langle p, f, \overline{Q} \rangle$ consists of elements in the image of $p$ and elements in the image of $\sigma \overline{Q}$, which are of the form $Q^{10}(y_4') + \xi_4^2Q^6(y_4')$. However, there are no indecomposables in the image of $p$ and no indecomposables in the dual Steenrod algebra in degree 5, and so the indeterminacy consists completely of decomposable elements. \qed

5.4 A secondary operation in the dual Steenrod algebra

Proposition 5.4.1. Suppose that $R$ is an $E_{12}$ $H$-algebra and $x \in \pi_2(R)$. Define the following classes:

\[
\begin{align*}
y_5 &= Q^3x \\
y_7 &= Q^5x \\
y_9 &= Q^7x \\
y_{13} &= Q^{11}x \\
y_8 &= Q^6x + x^4 \\
y_{10} &= Q^8x + x^2Q^4x \\
y_{12} &= Q^{10}x + (Q^4x)^2
\end{align*}
\]
Then there is an identity

\[ 0 = Q^{20}y_{10} + Q^{18}y_{12} + Q^{17}y_{13} + x^4(Q^{12}y_{10}) + y_2^2(Q^4x)^2 + y_2^2Q^5x + y_2^2Q^8Q^4x + (Q^9y_0)(Q^4x)^2 + (Q^{10}y_0)(Q^4x)^2 + y_2^2(Q^{11}Q^7x + Q^{10}Q^8x + x^4Q^6Q^4x) \]

Proof. The following table breaks this down term-by-term, substituting in the values of the \( y_i \).

\[
\begin{align*}
Q^{20}y_{10} &= Q^{20}Q^8x + Q^{20}(x^2Q^4x) \\
Q^{18}y_{12} &= Q^{18}Q^{10}x + Q^{18}((Q^4x)^2) \\
Q^{17}y_{13} &= Q^{17}Q^{11}x \\
x^4(Q^{12}y_{10}) &= x^4(Q^{12}Q^5x) + x^8Q^8Q^4x + x^4(Q^3x)^2Q^6Q^4x \\
y_2^2(Q^4x)^2 &= (Q^7x)^2(Q^4x)^2 \\
y_2^2Q^5x &= (Q^5x)^2Q^5x \\
y_2^2Q^8Q^4x &= (Q^6x)^2Q^6x + x^8Q^8Q^4x \\
(Q^9y_0)(Q^4x)^2 &= (Q^9Q^7x)(Q^4x)^2 \\
(Q^{10}y_0)(Q^4x)^2 &= (Q^{10}Q^6x)(Q^4x)^2 \\
y_2^2(Q^{11}Q^7x) &= (Q^3x)^2(Q^{11}Q^7x) \\
y_2^2(Q^{10}Q^8x) &= (Q^3x)^2Q^{10}Q^8x \\
y_2^2(x^4Q^6Q^4x) &= (Q^3x)^2(x^4Q^6Q^4x)
\end{align*}
\]

The reader who is interested in ensuring that these cancel is encouraged to do so with the aid of a pen. To assist this, we list the following needed identities deduced from the Cartan formula, Adem relations, and instability relations where appropriate.

\[
\begin{align*}
Q^{20}Q^8x &= Q^{18}Q^{10}x + Q^{17}Q^{11}x \\
Q^{20}(x^2Q^4x) &= x^4Q^{16}Q^8x + (Q^3x)^2Q^{14}Q^4x + (Q^4x)^2Q^{12}Q^4x + (Q^5x)^2Q^{10}Q^4x + (Q^7x)^2(Q^4x)^2 \\
Q^{18}((Q^4x)^2) &= 0 \\
x^4Q^{16}Q^4x &= x^4Q^{12}Q^8x \\
(Q^3x)^2Q^{14}Q^4x &= (Q^3x)^2Q^{11}Q^7x + (Q^3x)^2Q^{10}Q^8x \\
(Q^4x)^2Q^{12}Q^4x &= (Q^4x)^2Q^{10}Q^6x + (Q^4x)^2Q^9Q^7x \\
(Q^5x)^2Q^{10}Q^4x &= (Q^5x)^2Q^8Q^5x
\end{align*}
\]

To apply \( Q^r \) to an element in degree \( s \), as well as make use of the Adem relations, Cartan formula, and instability relations, we require the presence of an \( E_n \)-algebra for \( n \geq r - s + 2 \). The greatest value of \( n \) required from the equations above is when we take \( Q^{20}y_{10} \), and in particular use additivity for \( Q^{20} \), which requires an \( E_{12} \)-algebra. \( \square \)

We can use this relation to build secondary operations.
Proposition 5.4.2. Suppose \( n \geq 12 \) and let \( R \) be an object of \( C^n \), corresponding to an \( E_n \)-algebra with an element \( x \in \pi_2(R) \), such that the classes \( y_i \) of Proposition 5.4.1 vanish. Then there is a secondary operation on \( x \) given by \( \langle x, Q \mapsto R \rangle \in \pi_{31}R \). The indeterminacy in this secondary operation consists of elements of the form

\[
Q^{20}y'_{11} + Q^{18}y'_{13} + Q^{17}y'_{14}
\]

and decomposables. This secondary operation is preserved by the forgetful functors \( C^m \to C^n \) for \( m > n \).

Proof. For any \( n \geq 12 \), Proposition 5.4.1 describes a relation between homotopy operations, in the form of a homotopy commutative diagram of \( E_n \)-algebras

\[
\begin{array}{ccc}
\mathbb{P}^E_{E_n}(x, z_{30}) & \xrightarrow{R} & \mathbb{P}^E_{E_n}(x, y_5, y_6, y_7, y_9, y_{10}, y_{12}) \\
\varepsilon & & \downarrow Q \\
\mathbb{E}^E_{E_n}(x) & \xrightarrow{Q} & \mathbb{P}^E_{E_n}(x)
\end{array}
\]

adjoint to a commutative diagram of \( E_n \)-algebras under \( \mathbb{P}^E_{E_n}(x) \) of the form

\[
\begin{array}{ccc}
\mathbb{P}^E_{E_n}(x, z_{30}) & \xrightarrow{R} & \mathbb{P}^E_{E_n}(x, y_5, y_6, y_7, y_9, y_{10}, y_{12}) \\
\varepsilon & & \downarrow Q \\
\mathbb{E}^E_{E_n}(x) & \xrightarrow{Q} & \mathbb{P}^E_{E_n}(x)
\end{array}
\]

Here the maps \( Q \) and \( R \) are defined by the equations of Proposition 5.4.1. The map \( \varepsilon \) is a map of augmented objects, the domain by the map sending \( z_{30} \) to 0 and the range by the map sending all \( y_i \) to zero. In particular, the homotopy commutativity of the above diagrams shows that there exists a tethering \( Q \mapsto R \) in the category \( C^n \).

The indeterminacy in this secondary operation consists of elements in the image of the suspended operation

\[
\sigma R: \mathbb{P}^E_{E_n}(x, z_{31}) \to \mathbb{P}^E_{E_n}(x, y_6', y_8', y_{10}', y_{14}', y_9', y_{11}', y_{12}').
\]

Proposition 2.5.13 implies that \( \sigma R \) is given by

\[
z_{31} \mapsto Q^{20}y'_{11} + Q^{18}y'_{13} + Q^{17}y'_{14} + x^4(Q^{12}y'_{11}) + (Q^9y_6'(Q^4x)^2 + (Q^{10}y_6')(Q^4x)^2,
\]

since the other terms involve binary products that map to zero. However, the terms other than \( Q^{20}y'_{11} + Q^{18}y'_{13} + Q^{17}y'_{14} \) always take decomposable values. \( \square \)

Proposition 5.4.3. In the 2-primary dual Steenrod algebra

\[
H_2H \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots].
\]

viewed as the homotopy of the \( H \)-algebra \( H \wedge H \), the bracket \( \langle \xi_1^2, Q \rangle \) is defined, and the indeterminacy is zero mod decomposables.
Proof. The Cartan formula for Dyer–Lashof operations immediately implies that $Q^{2k+1}(\xi_1^2) = 0$ for all $k$. The remaining identities
\[
0 = Q^6\xi_1^2 + \xi_1^8 \\
0 = Q^8\xi_1^2 + \xi_1^4Q^4\xi_1^2 \\
0 = Q^{10}\xi_1^2 + (Q^4\xi_1^2)^2
\]
were determined in Proposition 5.2.2. Therefore, $\xi_1^2 Q = 0$ and the bracket $\langle \xi_1^2, Q, R \rangle$ is defined.

We now consider the indeterminacy. The indeterminacy is generated by adding the results of degree-29 homotopy operations applied to $\xi_1^2$, Dyer–Lashof operations applied to elements in degrees 11, 13, and 14, and decomposables. Proposition 2.5.12 showed that all nonconstant homotopy operations are generated by multiplication, addition, and the operations $Q^n$, all of which preserve decomposables. The dual Steenrod algebra contains no indecomposables in degrees 11, 13, and 14, and so any operation applied to such an element is decomposable. □

The operations $Q$ and $R$, while complex, can be related to simpler operations using the following diagram.

**Proposition 5.4.4.** Consider the maps

\[
\begin{align*}
\mu & : \mathbb{P}_H(x, y_5, y_7, y_9, y_{10}, y_{12}) \to \mathbb{P}_H(x, y_4) \\
\nu & : \mathbb{P}_H(x, z_{30}) \to \mathbb{P}_H(x, z_{14}) \\
\alpha & : \mathbb{P}_H(x, z_{30}) \to \mathbb{P}_H(x, z_{15}) \\
\beta & : \mathbb{P}_H(x, z_{15}) \to \mathbb{P}_H(x, y_4)
\end{align*}
\]

of augmented objects, defined by the identities

\[
\begin{align*}
\mu(y_i) &= 0 \text{ for } i \neq 10 \\
\mu(y_{10}) &= Q^6y_4 \\
\nu(z_{30}) &= Q^{16}z_{14} \\
\alpha(z_{30}) &= (z_{15})^2 \\
\beta(z_{15}) &= Q^3zQ^6y_4
\end{align*}
\]

Then there is an identity $\mu R = Q\nu + \beta\alpha$ and a homotopy commutative diagram in $C$ of the form

\[
\begin{array}{ccc}
\mathbb{P}_H(x, y_i) & \xrightarrow{Q} & \mathbb{P}_H(x) \\
\mu \downarrow & & \xi_1^2 \downarrow \\
\mathbb{P}_H(x, y_4) & \xrightarrow{f} & H \wedge MU \\
\end{array}
\]

where $f$ and $Q$ are from Theorem 5.3.1.
Proof. It is classical that the map \( p : H_2MU \to H_2H \) takes \( b_1 \) to \( \xi_1^2 \), making the right-hand triangle commute.

To verify that the map \( \mu \) makes the square diagram commute in the homotopy category, we need to know that the Dyer–Lashof operations on \( b_1 \) satisfy

\[
0 = Q^3b_1 \\
0 = Q^5b_1 \\
0 = Q^7b_1 \\
0 = Q^{11}b_1 \\
0 = Q^6b_1 + b_1^4 \\
0 = Q^{10}b_1 + (Q^4b_1)^2 \\
Q^6b_2 = Q^8b_1 + b_1^2Q^4b_1
\]

The odd operations vanish automatically because \( H_*MU \) is concentrated in even degrees, and the remaining three identities were proven in Proposition 5.1.2.

Finally we need to verify the identity \( Q\nu = \beta\alpha + \mu R \). Using the definition of \( \mu \) and the formula from Proposition 5.4.1 for \( R \), we find that

\[
\mu(R(z_{30})) = Q^{20}Q^6y_4 + x^4Q^{12}Q^6y_4
\]

In the Adem relation \( Q^{20}Q^6 = Q^{18}Q^{10} + Q^{14}Q^{12} + Q^{13}Q^{13} \), the last two terms automatically vanish on classes in degree four. Therefore, we can continue to simplify, finding

\[
\mu(R(z_{30})) = Q^{16}Q^{10}y_4 + x^4Q^{12}Q^6y_4 \\
= Q^{16}Q^{10}y_4 + Q^{16}(x^2Q^6y_4) + (Q^3x)^2(Q^6y_4)^2 \\
= Q^{16}(Q^{16}y_4 + x^2Q^6y_4) + Q^3xQ^6y_4 + (Q^3x)^2(Q^6y_4)^2 \\
= Q^{16}(\overline{Q}(z_{14})) + \beta(z_{15})^2 \\
= Q\nu(z_{30}) + \beta\alpha(z_{30}),
\]

as desired. \(\square\)

**Corollary 5.4.5.** In the dual Steenrod algebra, any element in the bracket \( \langle \xi_1^2, Q, R \rangle \) is congruent to \( \xi_5 \) mod decomposables.

*Proof.* Multiple applications of Proposition 2.3.5 and Proposition 2.4.3 give us the following string of identities, since all the intermediate terms are defined:

\[
\langle \xi_1^2, Q, R \rangle = \langle pb_1, Q, R \rangle \\
\subset \langle p, b_1Q, R \rangle \\
= \langle p, f\mu, R \rangle \\
\supset \langle p, f, \mu R \rangle \\
= \langle p, f, Q\nu + \beta\alpha \rangle \\
\subset \langle p, f, \overline{Q}\nu \rangle + \langle p, f, \beta\alpha \rangle \\
\supset \langle p, f, \overline{Q}\nu \rangle + \langle p, f, \beta\alpha \rangle.
\]

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By Theorem 5.3.1, we have
\[ \langle p, f, Q \rangle (v_{z_{30}}) = Q^{16}(\langle p, f, Q \rangle (z_{14})) \equiv Q^{16} \xi_4 \equiv \xi_5 \]
mod decomposables. On the other hand,
\[ \langle p, f, \beta \rangle (z_{30}) = (\langle p, f, \beta \rangle (z_{15}))^2 \]
which is automatically decomposable. Therefore, \( \langle \xi_1^2, Q, R \rangle \) contains \( \xi_5 \) mod decomposables.

We now consider the local maxima for indeterminacy in this chain of relations: the brackets \( \langle p, h_1Q, R \rangle \) and \( \langle p, f, Q \rangle + \langle p, f, \beta \alpha \rangle \). The sum total of the indeterminacy in these brackets consists of three types of elements. The first are elements in the image of \( H_*MU \to H_*H \), all of which are decomposable. The second are elements in the image of \( \sigma R \), which (as in Proposition 5.4.2) consists of multiples of Dyer–Lashof operations applied to elements in degrees 11, 13, and 14, which contain no indecomposables. The third consist of elements in the image of \( \sigma(Q) \) or \( \sigma(\beta \alpha) \), both of which are multiples of Dyer–Lashof operations applied to classes in degree 5, which contains no indecomposables. Therefore, the indeterminacy is zero mod decomposables.

**Theorem 5.4.6.** Suppose that \( n \geq 12 \) and \( R \) is an \( E_n \) ring spectrum with a map \( g : R \to H \) and an element \( x \in H_2(R) \) such that \( g(x) = \xi_1^2 \) in \( H_2H \). If the element \( x \) makes the classes \( y_i \) of Proposition 5.4.1 zero, then the map \( H_{31}R \to H_{31}H \) has \( \xi_5 \) in its image mod decomposables.

In particular, if \( H_*R \to H_*H \) is injective through degree 13, this result holds.

**Proof.** Under these conditions, \( H \wedge R \to H \wedge H \) is a map of \( E_n \)–algebras, and (up to equivalence) the map \( \mathcal{E}_H^R(x) \to H \wedge H \) lifts to a map \( \mathcal{E}_H^R(x) \to H \wedge R \). Thus, \( H \wedge R \to H \wedge H \) can be lifted to a map in \( C^n \) which, on homotopy groups, gives the map \( g : H_*R \to H_*H \).

Then the secondary operation \( (x, Q, R) \) is defined and the map \( g \) carries \( \langle x, Q, R \rangle \) into a subset of \( \langle \xi_1^2, Q, R \rangle \), all of whose elements are congruent to \( \xi_5 \) mod decomposables.

**5.5 Application to the Brown–Peterson spectrum**

Using Theorem 5.4.6, we can now exclude the existence of \( E_n \)-algebra structures on spectra related to the Brown–Peterson spectrum. We first recall the homology of the Brown–Peterson spectrum, dual to the cohomology described in [BP66].

**Proposition 5.5.1.** The Brown–Peterson spectrum \( BP \) is connective, with \( \pi_0BP \cong \mathbb{Z}(2) \).

The map \( BP \to \mathbb{H}F_2 \) induces an inclusion \( H_*BP \hookrightarrow H_*\mathbb{H}F_2 \) whose image is the subalgebra \( \mathbb{F}_2[\xi_1, \xi_2, \ldots] \subset \mathbb{F}_2[\xi_1, \xi_2, \ldots] \) of the dual Steenrod algebra. The image in positive degrees consists entirely of decomposables.
Similarly, we have truncated Brown–Peterson spectra $BP(k)$ and their generalized versions.

**Proposition 5.5.2 ([LN14, 4.3]).** Any generalized truncated Brown–Peterson spectrum $BP(k)$ is connective, with $\pi_0 BP(k) = \mathbb{Z}_{(2)}$. The map $BP(k) \rightarrow H^2 F_2$ induces an inclusion $H_* BP \hookrightarrow H_* H^2 F_2$ whose image is the subalgebra

$$\mathbb{F}_2[\xi_1^2, \xi_2^2, \ldots, \xi_{k+1}^2, \xi_{k+2}, \xi_{k+3}, \ldots] \subset \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

of the dual Steenrod algebra. The image in positive degrees consists entirely of decomposables until dimension $2^{k+2} - 2$.

In particular, the element $\xi_5$ is not in the image mod decomposables for $k \geq 4$. These spectra are also of finite type, and their 2-adic completions have the same homology groups.

By considering the cohomology in degree zero, we find that there is a unique nontrivial map of spectra $BP \rightarrow H^2 F_2$, and similarly for $BP(k)$. (At odd primes, this map is unique up to scalar.) As $E_n$-algebras have Postnikov towers, there is the following consequence.

**Corollary 5.5.3.** If $BP$ or $BP(k)$ admits the structure of an $E_n$-algebra, then the unique nontrivial map to $H^2 F_2$ lifts to a map of $E_n$-algebras.

We can now apply Theorem 5.4.6.

**Theorem 5.5.4.** The 2-local Brown–Peterson spectrum $BP$, the (generalized) truncated Brown–Peterson spectra $BP(k)$ for $k \geq 4$, and their 2-adic completions do not admit the structure of $E_n$-algebras for any $12 \leq n \leq \infty$.

**Remark 5.5.5.** The above results can also be applied to appropriate truncations in the Postnikov tower for $BP$.

## A Power operations in the Lazard ring

In this section we will extend Johnson–Noel’s proof of Theorem 4.1.4 to a proof that works in torsion-free quotients of the Lazard ring. The following calculations are specialized to the prime 2.

The power operation $P$ of Section 4.1 takes the form of a natural transformation

$$P: MU^{2n}(X) \rightarrow MU^{4n}(X \times B\Sigma_2) \hookrightarrow MU^{4n}(X)[[\alpha]]/[2]_F(\alpha).$$

Writing the 2-series as

$$[2]_F(\alpha) = \alpha \cdot (2)_F(\alpha),$$

we have the following properties.

- The identity $P(\alpha u) = P(\alpha)P(u)$ holds.
- The identity $P(u) \equiv u^2$ holds mod $\alpha$.  

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• The identity \( P(u + v) \equiv P(u) + P(v) \) holds mod \( \langle 2 \rangle_F(a) \). In particular, \( P \) becomes a ring homomorphism in this quotient.

• On the canonical orientation \( x \in \overline{MU}^2(\mathbb{CP}^\infty) \), we have \( P(x) = x(x + \alpha) \).

Let \( L = MU^* \) be the Lazard ring, and define \( g(x, \alpha) = x(x + \alpha) \) in the power series ring \( L[[x, \alpha]] \). Applying the identities for \( P \) to the spaces \( (\mathbb{CP}^\infty)^n \) and the natural maps between them, we deduce the following.

**Proposition A.0.1.** The map \( P \) induces a ring homomorphism

\[
\Psi : L \to L[[\alpha]]/\langle 2 \rangle_F(\alpha)
\]

and the power series \( g(x, \alpha) \) defines an isogeny \( F \to \Psi^* F \):

\[
g(x, \alpha) + \Psi^* F g(y, \alpha) \equiv g(x + F y, \alpha). \tag{A.1}
\]

The rings \( L \) and \( L[[\alpha]]/\langle 2 \rangle_F(\alpha) \) are torsion-free, and so the formal group laws \( F \) and \( \Psi^* F \) have logarithms:

\[
\ell_F(x) = \sum_{n} \frac{\mathbb{CP}^{n-1} x^n}{n} \quad \ell_{\Psi^* F}(x) = \sum_{n} \frac{\Psi(\mathbb{CP}^{n-1}) x^n}{n}
\]

By choosing any lifts of \( \Psi(\mathbb{CP}^n) \) to \( L[[\alpha]] \), we can view these formulas as defining power series \( \ell_F'(x) \in L[[x]] \) and \( \ell_{\Psi^* F}'(x) \in L[[x, \alpha]] \).

Taking derivatives of (A.1) with respect to \( y \) and evaluating at \( y = 0 \), we find

\[
\frac{g'(0)}{(\ell_{\Psi^* F})'(x)} \equiv \frac{g'(x)}{(\ell_F)'(x)}
\]

in \( L[[x, \alpha]]/\langle 2 \rangle_F(\alpha) \), and thus

\[
g'(x)(\ell_{\Psi^* F}')(g(x)) = \alpha \cdot (\ell_F)'(x) + h(x, \alpha) \cdot \langle 2 \rangle_F(\alpha) \tag{A.2}
\]

for some power series \( h(x, \alpha) \in L[[x, \alpha]] \).

We now substitute \( x = ay \) and observe that

\[
g(ay) = \alpha^2 k(y, \alpha),
\]

where the power series \( k(y, \alpha) \) has the form \( y + O(y^2) \). Hence, there a composition inverse: a series \( k^{-1}(y, \alpha) \) of the same form such that \( k(k^{-1}(y, \alpha), \alpha) = y \).

Substituting \( x = ay \) in to (A.2), we obtain an identity

\[
\alpha k'(y, \alpha)(\ell_{\Psi^* F}')(\alpha^2 k(y, \alpha)) = \alpha \cdot (\ell_F)'(ay) + h(ay, \alpha) \cdot \langle 2 \rangle_F(\alpha)
\]

which can be simplified to the statement

\[
(\ell_{\Psi^* F}')(\alpha^2 y) = (\ell_F)'(ak^{-1}(y, \alpha))(k^{-1})'(y, \alpha) + \tilde{h}(y, \alpha) \cdot \langle 2 \rangle_F(\alpha)
\]
for some series \( \hat{h}(y, \alpha) \in L[y, \alpha] \). If we write \( f_n(\alpha) \) for the coefficient of \( y^n \) in \((\ell_F)'(\alpha^{-1}(y, \alpha))(k^{-1})'(y, \alpha)\), we then find that

\[
\Psi(\mathbb{CP}^n) \alpha^{2n} = f_n(\alpha) + \hat{h}_n(\alpha) \cdot (2)_F(\alpha)
\]

for some series \( \hat{h}_n(\alpha) \). If \( f : L \to S \) is any ring homomorphism, there is a degree-2n polynomial \( h_n(\alpha) \in S[\alpha] \) such that

\[
f_n(\alpha) - h_n(\alpha) \cdot (2)_F(\alpha) \equiv (\mathbb{CP}^n)^2
\]
in \( S[[\alpha]]/(\alpha^{2n+1}) \). If 2 is not a zero divisor in the ring \( S \), this determines \( h_n(\alpha) \) uniquely and so it can be calculated in \( S \). We deduce that

\[
f(P(\mathbb{CP}^n)) \equiv \alpha^{-2n} \left( f_n(\alpha) - h_n(\alpha) \cdot (2)_F(\alpha) \right)
\]
in \( S[[\alpha]]/[2]_F(\alpha) \).

In particular, we may take \( S = \mathbb{Z}[[v_3]]/(v_3^2) \), which has logarithm \( x + \frac{v_3}{2} x^8 \). We can then expand out the definitions in this ring.

\[
x + y = x + y + \frac{v_3}{2} (x^8 + y^8 - (x + y)^8)
\]

\[
(2)_F(\alpha) = 2 - 127v_3\alpha^7
\]

\[
(\ell_F)'(x) = 1 + 4v_3x^7
\]

\[
g(x, \alpha) = ax + x^2 + \frac{v_3}{2} (a^8 + x^8 - (a + x)^8)
\]

\[
= ax + (1 - 4v_3\alpha^7)x^2 - 14v_3\alpha^6x^3 + O(x^4)
\]

\[
k(y, \alpha) = y + (1 - 4v_3\alpha^7)y^2 - 14v_3\alpha^7y^3 + O(y^4)
\]

\[
k^{-1}(y, \alpha) = y + (4v_3\alpha^7 - 1)y^2 + (2 - 2v_3\alpha^7)y^3 + O(y^4)
\]

\[
(\ell_F)'(ak^{-1}(y, \alpha)) = 1 + O(y^7)
\]

\[
(k^{-1})'(y, \alpha) = 1 + (8v_3\alpha^7 - 2)y + (6 - 6v_3\alpha^7)y^2 + O(y^3)
\]

\[
f_2(\alpha) = 6 - 6v_3\alpha^7
\]

\[
h_2(\alpha) = 3
\]

\[
f(P(\mathbb{CP}^2)) = 375v_3\alpha^3
\]

\[
\equiv v_3\alpha^3.
\]

Here the last congruence follows because, in the ring \( S[\alpha]/[2]_F(\alpha) \),

\[
2av_3 \equiv 127v_3^2\alpha^8 \equiv 0.
\]

Finally, \( P(\mathbb{CP}^2) \), in \( BP[\alpha]/[2]_F(\alpha) \) mod decomposables, can only involve \( v_3, v_4, \) and higher, so we find \( P(\mathbb{CP}^2) \equiv v_3\alpha^7 \) mod decomposables and higher-order terms in \( \alpha \) as desired.

**References**

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