

$$\frac{z^2 + z t_0 + t_0^2 - (t_1^2 + t_1 t_0 + t_0^2)}{z - t_1} = z + t_0 + t_1 = 0$$

few colinear pts. Taking inverse we see that the group law for this curve is the additive grp. $t_0 + t_1$. "

ex $C: y^2 - xy = x^3$ or $s - st = t^3$ or
 $s = \frac{t^3}{1-t}$ with $s = \frac{1}{y}$ and $t = \frac{x}{y}$.

$H^0(C, \Omega^1) = 1$ -dim' l gen. by

$$\varphi = \frac{dt}{1-t} = \frac{ds}{3t^2}$$

$$C \times C \xrightarrow{\mu} C$$

$$H^0(C, \Omega^1) \oplus H^0(C, \Omega^1) \xleftarrow{\mu^*} H^0(C, \Omega^1)$$

$$\pi_1^* \varphi + \pi_2^* \varphi \xleftarrow{\quad} \varphi$$

If we integrate to get the function

$$l(t) = \int \frac{dt}{1-t} = -\log(1-t),$$

this gives the equation

$$l(t_1 + t_2) = l(t_1) + l(t_2)$$

or, with $e(x) = e^{-1}(x)$,

$$t_1 +_c t_2 = e(\ell(t_1) + \ell(t_2))$$

$$= 1 - (1-t_1)(1-t_2) = t_1 + t_2 - t_1 t_2$$

— the multiplicative group. "

A formal group law over a ring R is a power series

$$x +_F y = F(x, y) \in R[[x, y]]$$

that satisfies

$$\text{(unital)} \quad x +_F 0 = 0 +_F x = x$$

$$\text{(commutative)} \quad x +_F y = y +_F x$$

$$\text{(associative)} \quad (x +_F y) +_F z = x +_F (y +_F z)$$

Let φ be the unique differential form which is invariant w.r.t. to F , i.e.

$$\varphi(x +_F y) = \varphi(x) + \varphi(y),$$

and which starts out as dx at

the origin. Then

$$\varphi(x) = \frac{dx}{\frac{\partial F}{\partial y}(x, 0)} \quad "$$

A formal group law is similar to a 1-dim'l mfd. plus a coordinate; wish to define formal group similar to 1-dim'l mfd. but without choice of coordinate.

Def A formal group over R is an R -algebra A together with an augmentation $\epsilon: A \rightarrow R$ s.t. A is I -adically complete ($I = \ker(\epsilon)$) and s.t. I/I^2 is projective of rk. 1 over R and together with a continuous ring-homomorphism

$$A \xrightarrow{\gamma} \hat{A} \hat{\otimes} A$$

that preserves the augmentation and that is unital, commutative, and associative. "

ex A formal group law gives a formal group

$$A = \mathbb{R}[[x]] \xrightarrow{\gamma} \mathbb{R}[[x, y]] = \hat{A} \otimes_{\mathbb{R}} A$$

$$x \mapsto F(x, y)$$

//

To connect w. topology, consider

E = even periodic, mult. coh. theory.

So $E^*(X)$ satisfies Mayer-Vietoris, $E^*(X)$ is a graded-commutative graded ring, $E^*(pt)$ is conc. in even degrees, $E^2(pt)$ is $f.g.$ over $E^0(pt)$ and

$$E^2(pt) \otimes_{E^0(pt)} E^{-2}(pt) \xrightarrow{\sim} E^0(pt)$$

Atiyah-Hirzebruch spectral sequence

$$H^*(X, E^*(pt)) \Rightarrow E^*(X)$$

Prop Suppose that E is even periodic and that $E_2(pt)$ is free of $rk. 1$ over $E^0(pt)$.

Then $E^0(\mathbb{C}P^\infty) \rightarrow E^0(\mathbb{C}P^1) = E_2(pt)$ is

onto. If $x \in E^0(\mathbb{C}P^1)$ maps to a generator of $E_2(pt)$, then $E^0(\mathbb{C}P^1) = \mathbb{Z}\langle x \rangle$ with $\mathbb{Z} = E^0(pt)$, and more generally,

$$E^0(\mathbb{C}P^1 \times \dots \times \mathbb{C}P^1) = \mathbb{Z}\langle x_1, \dots, x_k \rangle.$$

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1- The E_2 -term is conc. in even total degree, so the sp. seq. collapses. "

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \xrightarrow{\otimes} \mathbb{C}P^1$$

classifying \otimes -prod. of line bdl's.

$$\begin{array}{ccc} E^0(\mathbb{C}P^1 \times \mathbb{C}P^1) & \xleftarrow{\otimes^*} & E^0(\mathbb{C}P^1) \\ \text{"} & & \text{"} \\ A \hat{\otimes} A & \xleftarrow{\gamma} & A \end{array}$$

— a formal grp. (If $E_2(pt)$ is only projective of rk. 1, we can argue locally and still get a formal group.)

Question: Can we assoc. an E to every elliptic curve?

$$\begin{array}{ccc}
 E^0(\mathbb{C}P^n) & = & \text{ring of formal fets. on formal group } G / \mathbb{R} & \mathbb{P} \\
 \downarrow \varepsilon & & \downarrow & \downarrow \\
 E^0(\text{pt}) & = & \mathbb{R} & \mathbb{P}(0)
 \end{array}$$

\mathbb{I} = ideal of fets. vanishing at 0.

Consequence of Atiyah-Hirzebruch sp. seq.

$$\mathbb{I}/\mathbb{I}^2 \xrightarrow{\sim} \tilde{E}^0(\mathbb{C}P^n) = E^{-2}(\text{pt}) = E_2(\text{pt})$$

"
 cotangent space at $e \in G$ = space of invariant 1-forms on G

$$\mathbb{R} = E^0(\text{pt}) = E_0(\text{pt})$$

$$\omega = e^* \Omega_{G/\mathbb{R}}^1 \text{ invariant differentials}$$

$$E_{2n}(\text{pt}) = H^0(\text{Spec } \mathbb{R}, \omega^{\otimes n})$$

Def An elliptic cohomology theory is an even, periodic, multiplicative coh. th. E together with a generalized elliptic curve C over $\pi_0 E = E_0$ together with an iso. of formal groups over E_0 .

$$G_E \xrightarrow{\sim} \hat{C}$$

Would like invariant to distinguish formal group laws such as

$$\mathbb{G}_a(x, y) = x + y$$

$$\mathbb{G}_m(x, y) = x + y - xy$$

A formal group law F over \mathbb{Z}_p gives a group structure on $p\mathbb{Z}_p$. Can ask how many elem. of order p there are in this group. Write

$$[p](x) = \underbrace{x + \dots + x}_F \quad (p \text{ times})$$

For example

$$\mathbb{G}_a : [p](x) = px = 0 \implies x = 0$$

$$\mathbb{G}_m : [p](x) = 1 - (1-x)^p = 0$$

p solutions over $\mathbb{W}(\overline{\mathbb{F}_p})$.

Construct this invariant without reference to actual group:

$$\begin{array}{ccccc}
 pG & \longrightarrow & G & R[[x]] / [p](x) & \longleftarrow & R[[x]] & [p](x) \\
 \downarrow & & \downarrow \times p & \uparrow & & \uparrow & \mathbb{I} \\
 e & \longrightarrow & G & R & \longleftarrow & R[[x]] & x
 \end{array}$$

When is $R[[x]]/I_p(x)$ a free R -module, and what is its rank?

Suppose $R = \mathbb{F}_p$ -algebra.

Lemma Suppose $f: G_1 \rightarrow G_2$ is a homomorphism of formal group laws over R and that $f'(0) = 0$. Then $f(x) = g(x^p)$ for some $g \in R[[x]]$.

Pf Since f homo., $f'(0) = 0$ implies $f'(x) = 0$, for all x . (To prove this consider unique invariant diff. ω .)

$$f(x) = \sum a_n x^n$$

$$f'(x) = \sum n a_n x^{n-1} = 0 \implies$$

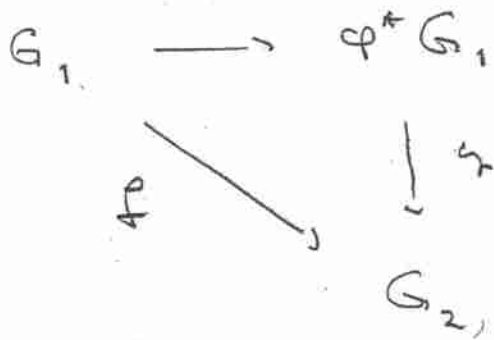
$$a_n = 0 \quad \text{if } p \nmid n.$$

$$\text{So } f(x) = g(x^p), \quad g(y) = \sum a_{np} y^n \quad //$$

$$\text{If } G(x, y) = \sum a_{ij} x^i y^j, \text{ let}$$

$$\varphi^* G(x, y) = \sum a_{ij}^p x^i y^j$$

Then $x \mapsto x^p$ defines a homomorphism $G \rightarrow \varphi^* G$. So lemma says that f factors

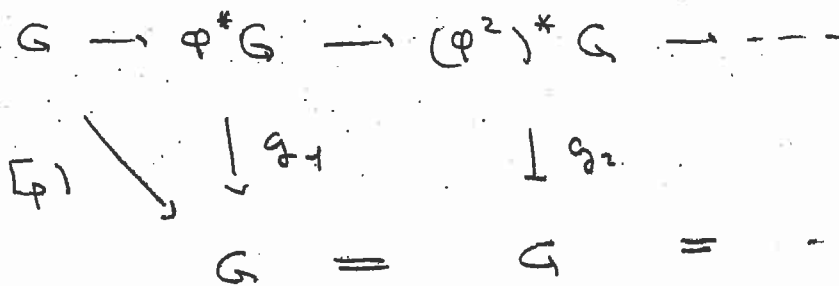


Cor If $[p]: G \rightarrow G$ is not zero, then

$$[p](x) = g(x^{p^h}), \quad g'(0) \neq 0 \in \mathbb{R}$$

$h \geq 1$

pf The lemma allows us to factor g_i as long as $g_i'(0) = 0$:



If $[p]$ is not zero, then we have $g_i'(0) \neq 0$, for some $i \geq 1$. //