$$\frac{z^2 + z t_0 + \cancel{t_0^2} - (t_1^2 + t_1 t_0 + \cancel{t_0^2})}{z - t_1} = z + t_0 + t_1 = 0$$

for colinear pts. Taking inverse we see that the group law for this curve is the additive grp. $t_0 + t_1$. //

ex C : $y^2 - xy = x^3$ or $s - st = t^3$ or

$s = \frac{t^3}{1-t}$ with $s = \frac{1}{y}$ and $t = \frac{x}{y}$.

$H^0(C, \Omega^1) =$ 1-dim'l gen. by

$$\varphi = \frac{dt}{1-t} = \frac{ds}{3t^2}$$

$$C \times C \xrightarrow{\mu} C$$

$$H^0(C, \Omega^1) \oplus H^0(C, \Omega^1) \xleftarrow{\mu^*} H^0(C, \Omega^1)$$

$$\pi_1^* \varphi + \pi_2^* \varphi \longleftarrow\!\!| \quad \varphi$$

If we integrate to get the function

$$\ell(t) = \int \frac{dt}{1-t} = -\log(1-t),$$

this gives the equation

$$\ell(t_1 \underset{c}{+} t_2) = \ell(t_1) + \ell(t_2)$$

or, with $e(x) = \ell^{-1}(x)$,

$$t_1 \underset{c}{+} t_2 = e(\ell(t_1) + \ell(t_2))$$

$$= 1 - (1-t_1)(1-t_2) = t_1 + t_2 - t_1 t_2$$

— the multiplicative group. "

A formal group law over a ring R is a power series

$$x \underset{F}{+} y = F(x,y) \in R[[x,y]]$$

that satisfies

(unital) $\quad x \underset{F}{+} 0 = 0 \underset{F}{+} x = x$

(commutative) $\quad x \underset{F}{+} y = y \underset{F}{+} x$

(associative) $\quad (x \underset{F}{+} y) \underset{F}{+} z = x \underset{F}{+} (y \underset{F}{+} z)$.

Let φ be the unique differential form which is invariant w.r.t. to F, i.e.

$$\varphi(x \underset{F}{+} y) = \varphi(x) + \varphi(y),$$

and which starts out as dx at

the origin. Then

$$\varphi(x) = \frac{dx}{\frac{\partial F}{\partial y}(x,0)}$$
//

A formal group law is similar to a 1-dim'l mfd. plus a coordinate; wish to define formal group similar to 1-dim'l mfd. but without choice of coordinate.

Def A formal group over R is an R-algebra A together with an augmentation $\varepsilon : A \to R$ s.t. A is I-adically complete ($I = \ker(\varepsilon)$) and s.t. I/I^2 is projective of rk. 1 over R and together with a continuous ring-homomorphism

$$A \xrightarrow{\psi} A \hat{\otimes} A$$

that preserves the augmentation and that is unital, commutative, and associative. //

ex A formal group law gives a formal group

$$\begin{aligned} A = R[[x]] &\xrightarrow{\psi} R[[x,y]] = A \hat{\otimes}_R A \\ x &\longmapsto F(x,y) \end{aligned}$$

//

To connect w. topology, consider

E = even periodic, mult. coh. theory.

So $E^*(X)$ satisfies Mayer-Victoris, $E^*(X)$ is a graded-commutative graded ring, $E^*(pt)$ is conc. in even degrees, $E^2(pt)$ is f.g. over $E^0(pt)$ and

$$E^2(pt) \underset{E^0(pt)}{\otimes} E^{-2}(pt) \xrightarrow{\sim} E^0(pt).$$

Atiyah-Hirzebruch spectral sequence

$$H^*(X, E^*(pt)) \Rightarrow E^*(X).$$

Prop Suppose that E is even periodic and that $E_2(pt)$ is free of rk. 1 over $E^0(pt)$. Then $E^0(\mathbb{C}P^\infty) \to E^0(\mathbb{C}P^1) = E_2(pt)$ is onto. If $x \in E^0(\mathbb{C}P^\infty)$ maps to a generator of $E_2(pt)$, then $E^0(\mathbb{C}P^\infty) = R[[x]]$ with $R = E^0(pt)$, and more generally,

$$E^0(\underbrace{\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty}_{k}) = R[[x_1, \ldots, x_k]].$$

pf The E_2-term is conc. in even total degree, so the sp. seq. collapses. //

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{\otimes} \mathbb{C}P^\infty$$

classifying $\otimes$-prod. of line bdls. ⇝

$$\begin{array}{ccc} E^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) & \xleftarrow{\otimes^*} & E^0(\mathbb{C}P^\infty) \\ \| & & \| \\ A \hat{\otimes} A & \xleftarrow{\psi} & A \end{array}$$

— a formal grp. (If $E_2(pt)$ is only projective of rk. 1, we can argue locally and still get a formal group.)

Question: Can we assoc. an E to every elliptic curve?

$$\begin{array}{ccccc}
E^0(\mathbb{C}P^\infty) & = & \text{ring of formal fcts. on formal group } G \,/\, R & & f \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow & & \downarrow \\
E^0(pt) & = & R & & f(0)
\end{array}$$

I = ideal of fcts. vanishing at 0.

Consequence of Atiyah-Hirzebruch sp. seq.

$$\begin{array}{ccc}
I/I^2 & \xrightarrow{\sim} & \tilde{E}^0(\mathbb{C}P^\infty) = E^{-2}(pt) = E_2(pt) \\
\| & & \\
\text{cotangent space at } e \in G & = & \text{space of invariant 1-forms on } G
\end{array}$$

$$R = E^0(pt) = E_0(pt)$$

$$\omega = e^* \Omega^1_{G/R} \quad \text{invariant differentials}$$

$$E_{2n}(pt) = H^0(\operatorname{Spec} R, \omega^{\otimes n}).$$

Def An elliptic cohomology theory is an even, periodic, multiplicative coh. th. E together with a generalized elliptic curve C over $\pi_0 E = E_0$ together with an iso. of formal grps over E_0

$$G_E \xrightarrow[\sim]{t} \hat{C}$$

 9/17/03.

Would like invariant to distinguish formal group laws such as

$$\mathbb{G}_a(x,y) = x+y$$

$$\mathbb{G}_m(x,y) = x+y-xy.$$

A formal group law F over $\mathbb{Z}_p$ gives a group structure on $p\mathbb{Z}_p$. Can ask how many elem. of order p there are in this group. Write

$$[p](x) = x \underset{F}{+} \cdots \underset{F}{+} x \qquad (p \text{ times})$$

For example

$$\mathbb{G}_a: \quad [p](x) = px = 0 \implies x = 0$$

$$\mathbb{G}_m: \quad [p](x) = 1-(1-x)^p = 0$$

p solutions over $W(\overline{\mathbb{F}}_p)$.

Construct this invariant without reference to actual group:

$$\begin{tikzcd}
{}_pG \arrow[r] \arrow[d] & G \arrow[d, "\times p"] \\
e \arrow[r] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
R[[x]]/[p](x) & R[[x]] \arrow[l] & {[p](x)} \\
R \arrow[u] & R[[x]] \arrow[l] \arrow[u] & x \arrow[u, mapsto]
\end{tikzcd}$$

When is $R[[x]]/[p](x)$ a free R-module, and what is its rank?

Suppose $R = \mathbb{F}_p$-algebra.

<u>Lemma</u> Suppose $f: G_1 \to G_2$ is a homomorphism of formal group laws over R and that $f'(0) = 0$. Then $f(x) = g(x^p)$ for some $g \in R[[x]]$.

<u>pf</u> Since f homo., $f'(0) = 0$ implies $f'(x) = 0$, for all x. (To prove this consider unique invariant diff. ω.)

$$f(x) = \sum a_n x^n$$

$$f'(x) = \sum n a_n x^{n-1} = 0 \Rightarrow$$

$$a_n = 0 \text{ if } p \nmid n.$$

So $f(x) = g(x^p)$, $g(y) = \sum a_{np} y^n$. //

If $G(x,y) = \sum a_{ij} x^i y^j$, let

$$\varphi^* G(x,y) = \sum a_{ij}^p x^i y^j$$

Then $x \mapsto x^p$ defines a homomorphism $G \to \varphi^* G$. So lemma says that f factors

$$\begin{tikzcd} G_1 \arrow[r] \arrow[dr, "f"'] & \varphi^* G_1 \arrow[d, "g"] \\ & G_2 , \end{tikzcd}$$

Cor If $[p] : G \to G$ is not zero, then

$$[p](x) = g(x^{p^h}) \, , \quad g'(0) \neq 0 \in R \, .$$
$$h \geq 1$$

pf The lemma allows us to factor g_i as long as $g_i'(0) = 0$;

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "{[p]}"'] & \varphi^* G \arrow[r] \arrow[d, "g_1"] & (\varphi^2)^* G \arrow[r] \arrow[d, "g_2"] & \cdots \\ & G \arrow[r, equal] & G \arrow[r, equal] & \cdots \end{tikzcd}$$

If $[p]$ is not zero, then we have $g_i'(0) \neq 0$, for some $i \geq 1$. //