

### Maps of stacks :

We assoc. to a Weierstrass eq. over  $\mathbb{R}$  a formal grp. law over  $\mathbb{R}$ , and to every change of coord. an iso. of formal group laws. This will give a map of stacks

$$M_{\text{Weier}}$$

$$\downarrow$$

$$M_{\text{FG}}$$

(Homotopy) pull-back of stacks :

$$\begin{array}{ccc} M_1 \times_M M_2 & \longrightarrow & M_1 \\ \downarrow & & \downarrow P_1 \\ M_2 & \xrightarrow{P_2} & M \end{array}$$

$$(M_1 \times_M M_2)(U) = \left\{ \begin{array}{l} a \in M_1(U) \quad P_1(a) \xrightarrow{t} P_2(b) \\ b \in M_2(U) \quad \text{in } M(U) \\ \text{commutative diagr.} \end{array} \right\}$$

Spaces  $\subset$  Stacks

~~App~~  $\subset$  Stacks

$$\text{Stack}(X; M) = M(X)$$

$(X_0, X_1)$  representable groupoid

$$M = \text{ass}(X_0, X_1)$$

$$\begin{array}{ccc}
 X_1 = X_0 \times_M X_0 & \longrightarrow & X_0 \\
 \downarrow & & \downarrow \\
 X_0 & \longrightarrow & M
 \end{array}$$

ex  $BG = \text{stack of principal } G\text{-bundles}$

$$\begin{array}{ccc}
 G & \longrightarrow & \text{pt} \\
 \downarrow & & \downarrow \\
 \text{pt} & \longrightarrow & BG
 \end{array}$$

Def A map  $M \rightarrow N$  of stacks on a site  $\mathcal{C}$  is representable if for every  $X \in \mathcal{C}$  and every map  $X \rightarrow M$ ,

the pull-back  $X \times_M M$  is representable. //

This allows us to extend local notions about maps in  $\mathcal{C}$  to notions in stacks.

For example, we say that  $M \rightarrow M$  is étale, if it is representable, and for every  $X \in \mathcal{A}ff$ , the base-change

$$X \times_M M \longrightarrow X$$

is an étale map in  $\mathcal{A}ff$ . //

ex Calc. pull-back

$$? \longrightarrow \text{Spec } \mathbb{Z}[a_1, \dots, a_6]$$

$$\downarrow \quad \quad \quad \downarrow \quad y^2 + a_1 x y + \dots$$

$$\text{Spec } \mathbb{Z}[c_4, c_6] \longrightarrow M_{\text{Weier}}$$

$$y^2 = x^3 + c_4 x + c_6$$

$$? \longrightarrow \text{Spec } \mathbb{A}[\lambda^{\pm 1}, r, s, t] \longrightarrow \text{Spec } \mathbb{Z}[a_1, \dots, a_6]$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \downarrow$$

$$\text{Spec } \mathbb{Z}[c_4, c_6] \longrightarrow \text{Spec } \mathbb{Z}[a_1, \dots, a_6] \longrightarrow M_{\text{Weier}}$$

so

$$\mathcal{Z} = \Gamma \otimes_{\mathbb{Z}[c_4, c_6]} \mathcal{Z}[c_4, c_6] = \mathcal{Z}[c_4, c_6][\lambda^{\pm 1}, r, s, t]$$

$$a_i \mapsto c_i \text{ resp. } 0.$$

$$y^2 = x^3 + c_4 x + c_6$$

change coord.

$$x \mapsto \lambda^{-2}(x+r)$$

$$y \mapsto \lambda^{-3}(y+sx+t)$$

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

$$(y+sx+t)^2 = (x+r)^3 + c_4(x+r) + c_6$$

$$a_1 \mapsto 2s$$

$$a_3 \mapsto 2t$$

$$a_2 \mapsto 3r - s^2$$

$$a_4 \mapsto c_4 + 3r^2 - 2st$$

$$a_6 \mapsto c_6 + c_4 r + r^3 - t^2$$

up to  
powers  
of  $\lambda$

Suppose we tensor with  $\mathbb{Z}[\frac{1}{6}]$

$$A \sim \mathbb{Z}[\frac{1}{6}][c_4, c_6, \lambda^{\pm 1}, r, s, t]$$

$$\swarrow \quad \searrow \sim$$
  
$$A[\lambda^{\pm 1}]$$

so over  $\mathbb{Z}[\frac{1}{6}]$

$$\text{Spec } A[\lambda^{\pm 1}]$$

faithfully flat

$$\text{Spec } A$$

+

faithfully flat

!

!

$$\text{Spec } \mathbb{Z}[\frac{1}{6}][c_4, c_6]$$

$$\mathcal{M}_{\text{Weier}}$$

faithfully flat



Sep. 29

More properties about stacks (on  $\mathcal{E}$ ).

Def. A map of stacks  $\mathcal{M}_0 \rightarrow \mathcal{Y}$  is representable if

$$\forall X \in \mathcal{E}, \quad \begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{M}_0 \\ \downarrow & \text{p.b.} & \downarrow \\ X & \longrightarrow & \mathcal{Y} \end{array} \implies \mathcal{Z} \in \mathcal{E}.$$

Prop: The diagonal map  $\mathcal{M}_0 \rightarrow \mathcal{M}_0 \times \mathcal{M}_0$  is representable

$$\text{iff } \forall X, Y \in \mathcal{E}, \quad \begin{array}{ccc} \mathcal{Z} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \mathcal{M}_0 \end{array} \implies \mathcal{Z} \in \mathcal{E}.$$

Pf:

$$\textcircled{\Rightarrow} \text{ given } \begin{array}{ccc} \mathcal{C} & \longrightarrow & X \\ \downarrow \text{ p.b.} & & \downarrow \\ Y & \longrightarrow & \mathcal{M}_0 \end{array}, \text{ can rearrange } \begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{M}_0 \\ \downarrow \text{ p.b.} & & \downarrow \\ X \times Y & \longrightarrow & \mathcal{M}_0 \times \mathcal{M}_0 \end{array}$$

this p.b. is equivalent

$$\implies \mathcal{C} \in \mathcal{E}.$$

$$\textcircled{\Leftarrow} \text{ given } \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{M}_0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{M}_0 \times \mathcal{M}_0 \end{array}, \text{ factor } \begin{array}{ccccc} \mathcal{A} & \longrightarrow & X & \longrightarrow & \mathcal{M}_0 \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X & \longrightarrow & \mathcal{M}_0 \times \mathcal{M}_0 \end{array}$$

$$\begin{array}{ccc} \mathcal{Z}_1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{M}_0 \end{array}$$

it's the same as p.b.

$$\implies \mathcal{Z}_1 \text{ a space}$$

$$\implies \mathcal{A} \text{ is p.b. of a space}$$

$$\text{then } \mathcal{A} = X \times_{X \times X} X \in \mathcal{E}.$$

//

$X = (X_0, X_1)$  a groupoid

$\mathcal{M}_X = \text{ans}(X_0, X_1)$

(in part.  $\text{id}_{X_0}$  is a canonical object here)

Prop:  $\mathcal{M}_X$  is spock.\* (\*)

\* i.e.  $\mathcal{M}_B \rightarrow \mathcal{M}_B \times \mathcal{M}_B$  is representable

(\*) property of  $\mathcal{C}$ :

given  $X$  and a covering  $\{U_\alpha \rightarrow X\}$

to give a map  $Y \rightarrow X$ , is equivalent to give

$$Y_\alpha \rightarrow U_\alpha + \text{iso } Y_\alpha|_{U_\alpha \cap U_\beta} \cong^{T_{\alpha\beta}} Y_\beta|_{U_\alpha \cap U_\beta}$$

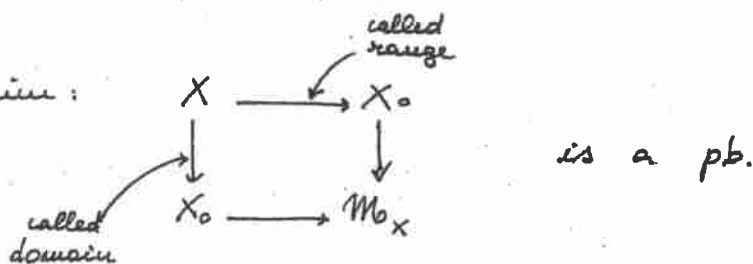
satisfying the cocycle condition

Equivalently:

$X \rightarrow \text{cat. of } Y \rightarrow X + \text{iso's } \begin{array}{c} Y_0 \xrightarrow{\sim} Y_1 \\ \searrow \quad \swarrow \\ X \end{array}$  is a stack.

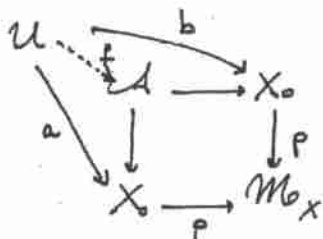
Pf.

1) claim:



is a pb.

suppose we have



$t$  represents  $(a, b, t)$  where  $t: p_a \xrightarrow{\sim} p_b$   
 $\cap$   
 $\mathcal{M}_X(U)$

to give  $t$  means, by definition, to give a covering  $U_\alpha \rightarrow U$  of  $U$  + iso's  $t_\alpha: p_{a_\alpha} \rightarrow p_{b_\alpha}$  compatible on overlaps.

NOTATION:

$$p_{a_\alpha} = p_a|_{U_\alpha}$$

Let's spell out the condition "compatible on overlaps".

1) claim:  $t_\alpha = t_\beta$  on  $U_\alpha \cap U_\beta$  (i.e. w.r.t. the identifications of these obj's)

consider  $p_{\alpha, \alpha} \xrightarrow{t_\alpha} p_{b, \alpha}$

$p_{\alpha, \beta} \xrightarrow{t_\beta} p_{b, \beta}$

and restrict to  $U_\alpha \cap U_\beta$ , get

$$\begin{array}{ccc} p_{\alpha, \alpha} & \longrightarrow & p_{b, \alpha} \\ \downarrow = & & \downarrow = \\ p_{\alpha, \beta} & \longrightarrow & p_{b, \beta} \end{array}$$

REM.  $t_\alpha: U_\alpha \rightarrow X$   
but used to mean  
the morphism  
 $p_{\alpha, \alpha} \xrightarrow{t_\alpha} p_{b, \alpha}$

id map  
b/c come  
from same  
obj's

Here  $t_\alpha$  can be really regarded as  $U_\alpha \rightarrow X$   
 $\Rightarrow$  they patch together to give a map  $U \rightarrow X$ .

Recall:

$\{U \rightarrow \mathcal{E}(U, X_0)\}$  objects  
 $\{U \rightarrow \mathcal{E}(U, X_1)\}$  maps

assoc. stack  $\leftarrow$  glue the local things to get  $U \rightarrow \mathcal{M}_0 X$ .

2) claim:  $A \rightarrow B \Rightarrow A \in \mathcal{E}$   
 $\downarrow p_b \downarrow$   
 $A \rightarrow \mathcal{M}_0 X$

to show  $A \in \mathcal{E}$  it suffices to find a covering  $\{U_\alpha \rightarrow A\}$   
and show  $A \in \mathcal{E}$ , where

$$\begin{array}{ccccc} c_\alpha & \longrightarrow & c_\beta & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ U_\alpha & \longrightarrow & A & \longrightarrow & \mathcal{M}_0 \end{array}$$

i.e. maps  
on covers s.t.  
comp. on overlaps

Given  $A \rightarrow \mathcal{M}_0$ , choose a covering  $\{U_\alpha \rightarrow A\}$  s.t.

$$\begin{array}{ccc} U_\alpha & \longrightarrow & X_0 \\ \downarrow \cong & & \downarrow \\ A & \longrightarrow & \mathcal{M}_0 \end{array}$$

(map to the associate stack  
factors through  $X_0$ , it's the def.)

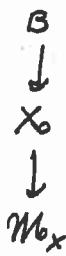
using claim 1, we reduce to the case

(A factors through  $X_0$ )

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ \Delta & \cdot & X_0 \longrightarrow \mathcal{M}_0 \end{array}$$

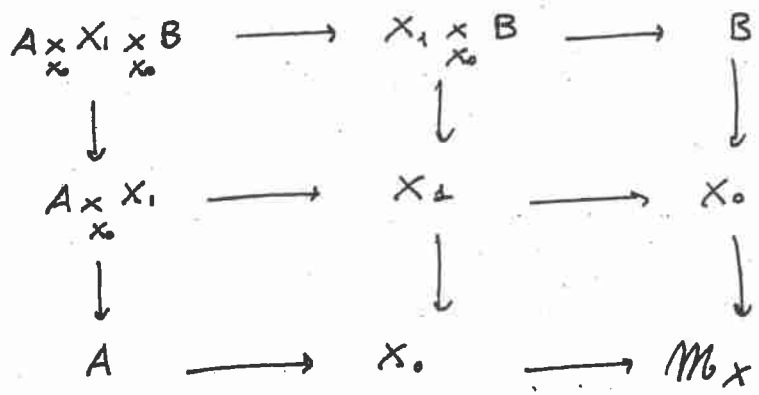


then we reduce to



$$A \rightarrow X_0 \rightarrow M_{B/X}$$

now form pb.'s in this diagram:



①  $X_0$  by def

②  $X_0 \times_{X_0} B$

③  $A \times_{X_0} X_1$

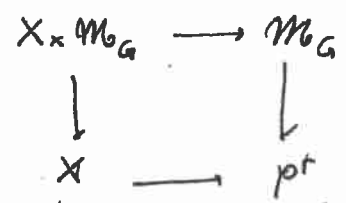
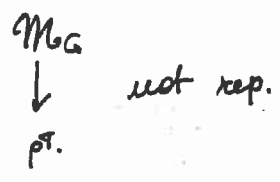
④  $\mathcal{U} = A \times_{X_0} X_1 \times_{X_0} B$

Examples:

↳ non rep.

Let  $M_{B/G} = \text{ass}(X, G)$

(*princ. G-bdlrs morph = iso between G bdlrs*)



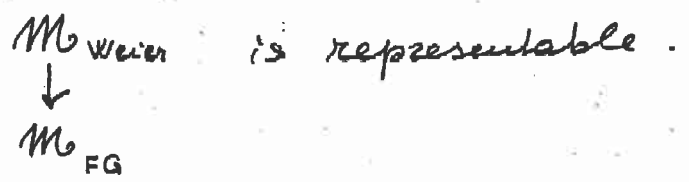
$X \times M_{B/G}$  is not a space if  $M_{B/G}$  is not.

$(X, G)$  can be both  $(*, G)$ ,  $(*, *)$

and  $\text{ass}(*, \bar{G}) = M_{B/G}$   
 $\downarrow$   
 $\text{ass}(*, *) = \text{pt}$

so maps of groupoids do not necessarily lead to rep. maps of stacks.

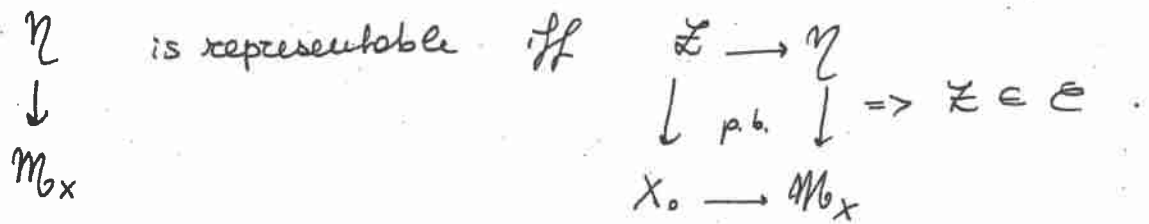
Thm: the map of stacks



Cor (of earlier prop.):

$X = (X_0, X_1)$ ,  $X_i$  a cover (flat, in the case of schemes)

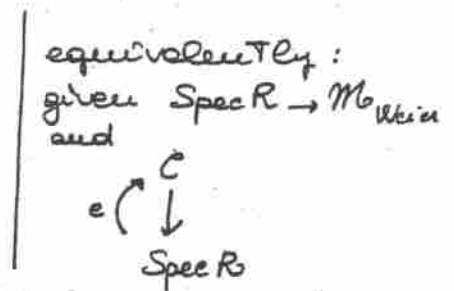
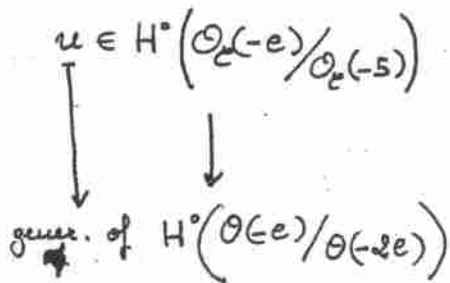
then



(i.e. only need to check on  $X_0 \rightarrow \mathcal{M}_{0,x}$ )

Lemma: Suppose  $\mathcal{E}/\mathbb{R}$  is locally given by Weierstrass equation.

Suppose



then

$\exists!$  functions

$$x \in H^0(\mathcal{O}_{\mathcal{E}}(2e))$$

i.e. has double pole in  $e$

$$y \in H^0(\mathcal{O}_{\mathcal{E}}(3e))$$

..... triple .....

$$\text{s.t. } x \equiv u^{-2} \pmod{\mathcal{O}_{\mathcal{E}}(e)} \quad (i)$$

$$y \equiv u^{-3} \pmod{\mathcal{O}_{\mathcal{E}}(2e)} \quad (ii)$$

$$\frac{x}{y} \equiv u \pmod{\mathcal{O}_{\mathcal{E}}(5e)} \quad (iii)$$

(i.e.  $\exists$  global functions over  $\mathcal{E}$ ,  $x$  and  $y$ , whose ratio is  $u$ )

then  $\mathcal{E}$  is given by  $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$

for  $a_1, \dots, a_6$

(sections of  $\mathcal{E}$  in degree 6  $\rightarrow$  linear relation between them)

i.e.  $\text{Spec } \mathbb{Z}[a_1, \dots, a_6]$  represents the functor "M (R). coord. mod"

Pf: the whole problem is local in  $\mathbb{R}$ , so we can assume  $\mathcal{C}$  given by some W. eq.  $y^2 + b_2 xy + b_3 y = \dots$   
 (locally we allow to scale  $x, y$ )

By scaling  $x, y$

$$\frac{x}{y} = u \pmod{\mathcal{O}(\geq 5)} \quad \left( \begin{array}{l} \text{i.e. } x = u^{-2} + \dots \\ y = u^{-3} + \dots \\ \text{but only up to this degree} \end{array} \right)$$

$z = \frac{x}{y}$  a local parameter near  $e$  ( $\infty$ )

$u$  other local par. near  $e$ , defined mod. terms of deg 5

so  $u$  looks like

$$u = c_1 z + c_2 z^2 + \dots + c_4 z^4 + \mathcal{O}(z^5)$$

$c_i \in \mathbb{R}^*$  b/c it has to be a local parameter

similarly

$$\begin{aligned} x = z^{-2} + \dots & \quad \begin{array}{l} \swarrow \text{in terms of } u \text{ (at least up to degree 5)} \\ = c_1^2 u^{-2} + ? u^{-1} + ? u^0 + ? u^1 + \dots \end{array} \\ y = z^{-3} + \dots & \quad = c_1^3 u^{-3} + \dots \end{aligned}$$

no sense

But conditions (i), (ii) are not satisfied

but locally, in the flat topology on  $\mathbb{R}$

$$\mathbb{R} \xrightarrow[\text{faithfully flat}]{} \mathbb{R}[t]_{t^5 = c_1}$$

locally we may therefore assume  $\mathcal{C}_1 = t^5$

$$\begin{array}{l} \text{replace } x \rightarrow c_1^2 x \\ y \rightarrow c_1^3 y \end{array} \quad \text{then } z \rightarrow c_1^{-1} z \Rightarrow z = u + \dots$$

$$\begin{array}{l} \text{scaling, } x \text{ looks like } u^{-2} + \dots \quad \Rightarrow \text{(i)} \\ y \text{ -- -- } u^{-3} + \dots \quad \Rightarrow \text{(ii)} \end{array}$$

but now  $z = \frac{x}{y}$  doesn't satisfy (ii)

but know that

$$z = u + c \cdot u^2 + \dots \quad \text{b/c (i)}$$

change  $y \rightarrow y + sx$

$$\frac{x}{y} \rightarrow \frac{x}{y+sx} = x - sx^2 + o(x^3)$$

$\exists! s$  ( $s=c$ ) s.t.  
the new  $x$  satisfies  
 $x = u + d.u^3 + \dots$

replacing  $x \rightarrow x + t$   
 $y \rightarrow y + x u^2$

$\exists! x$

$$z = u + m u^4 + \dots$$

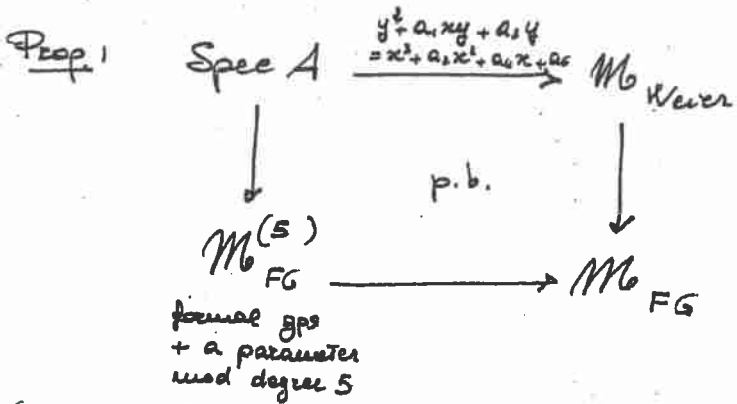
and with  $y \rightarrow y + z$

$$z \rightarrow z - t z^2 + \dots \quad \exists! t \text{ s.t. new } x = u + O(5)$$

$\Rightarrow \exists!$  choices s.t. (iii) satisfied.

Oct 1

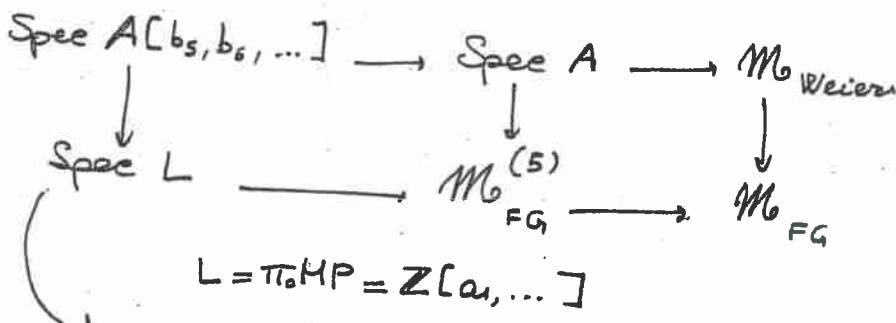
thru:  $\mathcal{M}_0_{\text{Weier}}$   
 $\downarrow$   
 $\mathcal{M}_0_{FG}$  is representable



where  
 $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_5]$

(it's the proposition proved last time)

Pf (thru)



in  $y + a_1xy + \dots$   
we have coordinates  
 $z = \frac{x}{y}$   
new coord.

Def. A map  $\eta$  is  $\left\{ \begin{array}{l} \text{flat} \\ \text{étale} \\ \text{a cover} \end{array} \right.$  if it is representable

and, for any  $U \rightarrow M_0$ ,  $W \rightarrow \eta$  the map  $W \rightarrow U$

$$\begin{array}{ccc} & \downarrow \text{pb.} & \downarrow \\ U & \rightarrow & M_0 \end{array}$$

is  $\left\{ \begin{array}{l} \text{flat} \\ \text{étale} \\ \text{a cover} \end{array} \right.$ .

We get a Grothendieck topology  $(M_0)_{\text{ét}}$ .

$p > 3$ ,  $M_{0, \text{Weier}} \otimes \mathbb{Z}/p$  reduce mod  $p$

$$\begin{array}{c} \text{Spec } \mathbb{Z}/p[a_6] \\ \downarrow y^2x^3 + x + a_6 \end{array}$$

claim: this is étale.

$$M_{0, \text{Weier}} \otimes \mathbb{Z}/p$$

Since  $M_{0, \text{Weier}} \otimes \mathbb{Z}/p = \text{ass}(\text{Spec } \mathbb{Z}/p[c_4, c_6], R[d^{*1}])$

$$? \longrightarrow \text{Spec } \mathbb{Z}/p[a_6]$$

$$\begin{array}{l} c_4 \rightarrow d^4 c_4 \\ c_6 \rightarrow d^6 c_6 \end{array}$$

$$\text{Spec } \mathbb{Z}/p[c_4, c_6] \xrightarrow{\text{is a cover}} M_{0, \text{Weier}} \otimes \mathbb{Z}/p$$

it suffices to check that this is étale

$$\text{Spec } \mathbb{Z}/p[a_6, d^{*1}] \longrightarrow \text{Spec } \mathbb{Z}/p[a_6]$$

$$\text{Spec } \mathbb{Z}/p[c_4, c_6, d^{*1}] \longrightarrow \text{Spec } \mathbb{Z}/p[c_4, c_6]$$

$$\text{Spec } \mathbb{Z}/p[c_4, c_6] \longrightarrow M_{0, \text{Weier}} \otimes \mathbb{Z}/p$$

to work out what this map is:

$$\begin{array}{cc} 0 & a_6 \\ \uparrow & \uparrow \\ c_4 & c_6 \end{array}$$

$$\text{so } \mathbb{Z}/p[c_4, c_6] \rightarrow \mathbb{Z}/p[a_6, \lambda^{\neq 1}]$$

$$c_6 \mapsto d^6 a_6$$

$$c_4 \mapsto \lambda^4$$

$$\mathbb{Z}/p[c_4, c_6] \longrightarrow \mathbb{Z}/p[a_6, \lambda^{\neq 1}]$$

$$\mathbb{Z}/p[c_4^{-1}, c_6]$$

add a 4<sup>th</sup> root of  $c_4$

$$\mathbb{Z}/p[a_6, \lambda^{\neq 1}] = \lambda^{-1} \mathbb{Z}/p[c_4, c_6][\lambda] / \lambda^4 = c_4$$

flat and unramified.

then:

- If a sheaf  $\mathcal{O}^{\text{Top}}$  on  $(\mathcal{M}_0^{\text{Weier}})_{\text{et}}$  with values in  $A_{\infty}$  ring spectra with the property that

$$\mathcal{O}(\text{Spec } R \xrightarrow{\mathcal{E}} \mathcal{M}_0^{\text{Weier}})$$

$\mathcal{E}$  = elliptic curve

is an even periodic cohomology theory  $E$  with  $\pi_0 E = R$  and  $FG = \text{completion of } E$ .

Rem:  $\pi_0 \mathcal{O}^{\text{Top}}$  is a presheaf of rings and also  $\pi_0 \mathcal{O}^{\text{Top}}$  a sheaf of functions. Similarly also  $\pi_{2m} \mathcal{O}^{\text{Top}} = \omega^m$ .

- Unique up to homotopy equivalence (of sheaves of  $A_{\infty}$  ring spectra)

Moreover  $\mathcal{O}^{\text{Top}}(\mathcal{M}_0^{\text{Weier}})$  is  $E_{\infty}$ .

Def:  $T_m f = (-1)$ -connected cover of  $\mathcal{O}^{\text{Top}}(\mathcal{M}_0^{\text{Weier}})$

(functorially; spectra  $\rightsquigarrow$  elliptic curve)

About the notions of  $A_{\infty}$ ,  $E_{\infty}$ .

DICTIONARY

Spectra	Abelian gps
$\wedge$ smash product (derived from smash product spaces)	$\otimes$ Tensor product
$\vee$ wedge product (coproduct of spectra)	$\oplus$ Whitney sum
$A_{\infty}$ ring spectrum	associative algebra $R$ with $R \otimes R \rightarrow R$ (1)
$E_{\infty}$ ring spectrum (symmetric algebra in cat of spectra)	commutative ring (e)
$S^0 \wedge E \cong E$	$\mathbb{Z} \otimes A = A$ (unit)
$X \wedge_R Y$	$M \otimes_R N$ (formally works the same)
$S^0 \rightarrow R$	$\mathbb{Z} \xrightarrow[\text{unit}]{1} R$

Notes:

(1) formulation of associativity  $\rightarrow$  diagrams involving triple  $\otimes$   
 $(A \otimes B) \otimes C \neq A \otimes (B \otimes C)$  but canonically isomorphic.  
 This becomes a deeper issue in topology. Was once a large problem ( $A_{\infty}$ : A for Associative, as for infinitely many conditions). But now we have a different way of setting up smash product ( $\wedge$  strictly associative)