Maps of stacks:

We assoc. to a Weierstrass eq. over R a formal grp. law over R, and to every change of coord. an iso. of formal group laws. This will give a map of stacks

$$\begin{array}{c} \mathcal{M}_{Weier} \\ \downarrow \\ \mathcal{M}_{FG} \end{array}.$$

(Homotopy) pull-back of stacks:

$$\begin{tikzcd} \mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2 \arrow[r] \arrow[d] & \mathcal{M}_1 \arrow[d, "p_1"] \\ \mathcal{M}_2 \arrow[r, "p_2"] & \mathcal{N} \end{tikzcd}$$

$$(\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2)(U) = \left\{ \begin{array}{ll} a \in \mathcal{M}_1(U) & p_1(a) \xrightarrow{t} p_2(b) \\ b \in \mathcal{M}_2(U) & \text{in } \mathcal{N}(U) \\ & + \\ \multicolumn{2}{c}{\text{commutative diagr.}} \end{array} \right\}.$$

Spaces $\subset$ Stacks

Aff $\subset$ Stacks

$$\mathrm{Stack}(X; \mathcal{M}) = \mathcal{M}(X).$$

(X_0, X_1) representable groupoid.

$\mathcal{M} = \mathrm{ass}(X_0, X_1)$.

$$\begin{tikzcd} X_1 = X_0 \times_{\mathcal{M}} X_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ X_0 \arrow[r] & \mathcal{M} \end{tikzcd}$$

ex BG = stack of principal G-bdle.

$$\begin{tikzcd} G \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & BG \end{tikzcd}$$

Def A map $\mathcal{N} \to \mathcal{M}$ of stacks on a site $\mathcal{C}$ is representable if for every $X \in \mathcal{C}$ and every map $X \to \mathcal{M}$, the pull-back $X \times_{\mathcal{M}} \mathcal{M}'$ is representable. //

This allows us to extend local notions about maps in $\mathcal{C}$ to notions in stacks. For example, we say that $\mathcal{M}' \to \mathcal{M}$ is étale, if it is representable, and for every $X \in \mathrm{Aff}$, the base-change

$$X \times_{\mathcal{M}} \mathcal{M}' \longrightarrow X$$

is an étale map in Aff. //

ex Calc. pull-back

$$\begin{array}{ccc} ? & \longrightarrow & \operatorname{Spec} \mathbb{Z}[a_1, \dots, a_6] \\ \downarrow & & \downarrow \; y^2 + a_1 xy + \cdots \\ \operatorname{Spec} \mathbb{Z}[c_4, c_6] & \longrightarrow & \mathcal{M}_{\text{Weier}} \end{array}$$

$$y^2 = x^3 + c_4 x + c_6$$

$$\begin{array}{ccccc} ? & \to & \operatorname{Spec} A[\lambda^{\pm 1}, r, s, t] & \to & \operatorname{Spec} \mathbb{Z}[a_1, \dots, a_6] \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} \mathbb{Z}[c_4, c_6] & \to & \operatorname{Spec} \mathbb{Z}[a_1, \dots, a_6] & \longrightarrow & \mathcal{M}_{\text{Weier}} \end{array}$$

So

$$? = \Gamma \underset{A}{\otimes} \mathbb{Z}[c_4, c_6] = \mathbb{Z}[c_4, c_6][\lambda^{\pm 1}, r, s, t]$$

$$a_i \longmapsto c_i \text{ resp. } 0.$$

$$y^2 = x^3 + c_4 x + c_6$$

change coord.

$$x \longmapsto \lambda^{-2}(x+r)$$

$$y \longmapsto \lambda^{-3}(y+sx+t). \qquad \rightsquigarrow$$

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

$$(y+sx+t)^2 = (x+r)^3 + c_4(x+r) + c_6$$

$$a_1 \longmapsto 2s$$

$$a_3 \longmapsto 2t$$

$$a_2 \longmapsto 3r - s^2$$

$$a_4 \longmapsto c_4 + 3r^2 - 2st$$

$$a_6 \longmapsto c_6 + c_4 r + r^3 - t^2$$

up to powers of λ

Suppose we tensor with $\mathbb{Z}[\tfrac{1}{6}]$

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] & & \mathbb{Z}[\tfrac{1}{6}][c_4, c_6, \lambda^{\pm 1}, r, s, t] \\
& A[\lambda^{\pm 1}] \arrow[ur, "\sim"'] &
\end{tikzcd}$$

so over $\mathbb{Z}[\tfrac{1}{6}]$

$$\begin{tikzcd}
\operatorname{Spec} A[\lambda^{\pm 1}] \arrow[r, "\text{faithfully flat}"] \arrow[d] & \operatorname{Spec} A \arrow[d, "\text{faithfully flat}"] \\
\operatorname{Spec} \mathbb{Z}[\tfrac{1}{6}][c_4, c_6] \arrow[r, "\text{faithfully flat}"'] & \mathcal{M}_{\text{Weier}}
\end{tikzcd}$$

+

Sep. 29

More properties about stacks (on $\mathcal{E}$).

Def. A map of stacks $\mathcal{M} \to \mathcal{N}$ is <u>representable</u> if

$\forall X \in \mathcal{E}$,
$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{M} \arrow[d] \\ X \arrow[r] & \mathcal{N} \end{tikzcd} \quad \Rightarrow Z \in \mathcal{E}.$$

Prop: the diagonal map $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable iff $\forall X, Y \in \mathcal{E}$,
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & \mathcal{M} \end{tikzcd} \quad \Rightarrow Z \in \mathcal{E}.$$

Pf:

($\Rightarrow$) given
$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pb.}"] & X \arrow[d] \\ Y \arrow[r] & \mathcal{M} \end{tikzcd}$$
, can rearrange
$$\begin{tikzcd} \mathcal{H} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{M} \arrow[d] \\ X \times Y \arrow[r] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$
this p.b. is equivalent

$\Rightarrow \mathcal{A} \in \mathcal{E}$.

($\Leftarrow$) given
$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ X \arrow[r] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$
, factor
$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}_1"] & \mathcal{M} \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X \arrow[r] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & \mathcal{M} \end{tikzcd}$$
it's the same as pb_1.

$\Rightarrow Z_1$ a space

$\Rightarrow \mathcal{A}$ is pb. of a space

then $\mathcal{A} = X \times_{X \times X} X \in \mathcal{E}$.

//

$X = (X_0, X_1)$ a groupoid

$\mathcal{M}_X = \mathrm{ass}(X_0, X_1)$

(in part. id_{X_0} is a canonical object here)

<u>Prop</u>: $\mathcal{M}_X$ is spock.* (*)

*i.e. $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable

(*) property of $\mathcal{C}$:
given X and a covering $\{U_\alpha \to X\}$
to give a map $Y \to X$, is equivalent to give
$Y_\alpha \to U_\alpha$ + iso $\; Y_\alpha\big|_{U_\alpha \cap U_\beta} \overset{\tau_{\alpha\beta}}{\cong} Y_\beta\big|_{U_\alpha \cap U_\beta}$

satisfying the cocycle condition

Equivalently:

$X \longmapsto$ cat. of $Y \to X$ + iso's
$$\begin{tikzcd} Y_0 \arrow[rr, "\sim"] \arrow[dr] & & Y_1 \arrow[dl] \\ & X & \end{tikzcd}$$
is a stack.

<u>Pf</u>.

1) claim:
$$\begin{tikzcd} X \arrow[r, "\text{called range}"] \arrow[d, "\text{called domain}"'] & X_0 \arrow[d] \\ X_0 \arrow[r] & \mathcal{M}_X \end{tikzcd}$$
is a pb.

suppose we have
$$\begin{tikzcd} U \arrow[dr, dotted, "f"] \arrow[drr, bend left, "b"] \arrow[ddr, "a"'] & & \\ & A \arrow[r] \arrow[d] & X_0 \arrow[d, "p"] \\ & X_0 \arrow[r, "p"'] & \mathcal{M}_X \end{tikzcd}$$

f represents (a, b, t) where
$t : pa \xrightarrow{\sim} pb$
$\in \mathcal{M}_X(U)$

to give t means, by definition, to give a covering $U_\alpha \to U$ of U + iso's $t_\alpha : pa_\alpha \longrightarrow pb_\alpha$ compatible on overlaps.

NOTATION: $pa_\alpha = pa\big|_{U_\alpha}$

Let's spell out the condition "compatible on overlaps".

1) claim: t_α "=" t_β on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ (i.e. w.r.t. the identifications of these obj's)

consider $pa_\alpha \xrightarrow{t_\alpha} pb_\alpha$

$pa_\beta \xrightarrow{t_\beta} pb_\beta$

and restrict to $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, get

$$\begin{tikzcd} pa_\alpha \arrow[r] \arrow[d, "="] & pb_\alpha \arrow[d, "="] \\ pa_\beta \arrow[r] & pb_\beta \end{tikzcd}$$

id map b/c come from same obj's

REM. $t_\alpha : \mathcal{U}_\alpha \to X$ but used to mean the morphism $pa_\alpha \xrightarrow{t_\alpha} pb_\alpha$

Here t_α can be really regarded as $\mathcal{U}_\alpha \to X$
$\Rightarrow$ they patch together to give a map $\mathcal{U} \to X_1$.

Recall:
$\left\{\begin{array}{ll} \mathcal{U} \to \mathcal{E}(\mathcal{U}, X_0) & \text{objects} \\ \mathcal{U} \to \mathcal{E}(\mathcal{U}, X_1) & \text{maps} \end{array}\right.$
assoc. stack $\leftarrow$ glue the local things to get $\mathcal{U} \to \mathcal{M}_X$.

2) claim:
$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & B \arrow[d] \\ A \arrow[r] & \mathcal{M}_X \end{tikzcd} \Rightarrow \mathcal{A} \in \mathcal{E}$$

to show $\mathcal{A} \in \mathcal{E}$ it suffices to find a covering $\{\mathcal{U}_\alpha \to A\}$ and show $\mathcal{A}_\alpha \in \mathcal{E}$, where

$$\begin{tikzcd} \mathcal{A}_\alpha \arrow[r] \arrow[d] & \mathcal{A} \arrow[r] \arrow[d] & B \arrow[d] \\ \mathcal{U}_\alpha \arrow[r] & A \arrow[r] & \mathcal{M} \end{tikzcd}$$

i.e. maps on covers s.T comp. on overlaps

Given $A \to \mathcal{M}$, choose a covering $\{\mathcal{U}_\alpha \to A\}$ s.T.

$$\begin{tikzcd} \mathcal{U}_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "//"] & X_0 \arrow[d] \\ A \arrow[r] & \mathcal{M} \end{tikzcd}$$

(map to the associate stack factors through X_0, it's the def.)

using claim 1, we reduce to the case

(A factors through X_0)

$$\begin{tikzcd} & B \arrow[d] \\ \mathcal{A} \quad . \quad X_0 \arrow[r] & \mathcal{M}_X \end{tikzcd}$$

then we reduce to

$$\begin{tikzcd} & & B \arrow[d] \\ & & X_0 \arrow[d] \\ A \arrow[r] & X_0 \arrow[r] & \mathcal{M}_X \end{tikzcd}$$

now form pb.'s in this diagram :

$$\begin{tikzcd} A \times_{X_0} X_1 \times_{X_0} B \arrow[r] \arrow[d] & X_1 \times_{X_0} B \arrow[r] \arrow[d] & B \arrow[d] \\ A \times_{X_0} X_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ A \arrow[r] & X_0 \arrow[r] & \mathcal{M}_X \end{tikzcd}$$

① X_1 by def

② $X_0 \times_{X_0} B$

③ $A \times_{X_0} X_1$

④ $\mathcal{A} = A \times_{X_0} X_1 \times_{X_0} B$

//

Examples :

1) non rep.

Let $\mathcal{M}_G = \mathrm{ass}((X,G))$ (princ. G-bdles morph = iso between G bdles)

$\mathcal{M}_G \to pt.$ not rep.

$$\begin{tikzcd} X \times \mathcal{M}_G \arrow[r] \arrow[d] & \mathcal{M}_G \arrow[d] \\ X \arrow[r] & pt \end{tikzcd}$$

$X \times \mathcal{M}_G$ is not a space if $\mathcal{M}_G$ is not.

(X, G) can be both $(*, G)$, $(*, *)$

and

$$\begin{tikzcd} \mathrm{ass}(*, \bar{G}) = \mathcal{M}_G \arrow[d] \\ \mathrm{ass}(*, *) = pt \end{tikzcd}$$

so maps of groupoids do not necessarily lead to rep. maps of stacks.

Thm: the map of stacks $\mathcal{M}_{\mathrm{Weier}} \to \mathcal{M}_{FG}$ is representable.

Cor (of earlier prop.):

$X = (X_0, X_1)$, $\begin{array}{c} X_1 \\ \downarrow \\ X \end{array}$ a cover (flat, in the case of schemes)

then

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ \mathcal{M}_X \end{array} \quad \text{is representable iff} \quad \begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & \mathcal{Y} \arrow[d] \\ X_0 \arrow[r] & \mathcal{M}_X \end{tikzcd} \Rightarrow \mathcal{Z} \in \mathcal{C}.$$

(i.e. only need to check on $X_0 \to \mathcal{M}_X$)

Lemma: Suppose $\mathcal{E}/R$ is locally given by Weierstrass equation.

Suppose

$$\begin{array}{ccc} u \in & H^0\left(\mathcal{O}_{\mathcal{E}}(-e)/\mathcal{O}_{\mathcal{E}}(-5)\right) & \\ \downarrow & \downarrow & \\ \text{gener. of} & H^0\left(\mathcal{O}(-e)/\mathcal{O}(-2e)\right) & \end{array}$$

equivalently: given $\operatorname{Spec} R \to \mathcal{M}_{Weier}$ and $e \left(\begin{array}{c} \mathcal{E} \\ \downarrow \\ \operatorname{Spec} R \end{array} \right.$

then

$\exists!$ functions

$x \in H^0(\mathcal{O}(2e))$ i.e. has double pole in e

$y \in H^0(\mathcal{O}(3e))$ . . . . . . triple . . . —

s.t.

$x \equiv u^{-2} \mod \mathcal{O}(e)$ (i)

$y \equiv u^{-3} \mod \mathcal{O}(2e)$ (ii)

$\frac{x}{y} \equiv u \mod \mathcal{O}(-5e)$ (iii)

(i.e. $\exists$ global functions over $\mathcal{E}$, x and y, whose ratio is u)

then $\mathcal{E}$ is given by $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$

for $!\ a_1, \ldots, a_6$ (sections of $\mathcal{E}$ in degree 6 $\Rightarrow$ linear relation between them)

i.e. $\operatorname{Spec} \mathbb{Z}[a_1 \ldots a_6]$ represents the functor "$\mathcal{M}_1(R)$, coord. mod"

Pf: the whole problem is local in R, so we can assume $\mathcal{E}$ given by some W. eq: $y^2 + b_1 xy + b_3 y = \dots$ (locally we allow to scale x, y)

By scaling x, y

$\frac{x}{y} = u \mod \mathcal{O}(-8e)$ $\left(\text{i.e. } \begin{array}{l} x = u^{-2} + \dots \\ y = u^{-3} + \dots \\ \text{but only up to this degree} \end{array} \right)$

$z = \frac{x}{y}$ a local parameter near e (∞)

u other local par. near e, defined mod. terms of deg 5

so u looks like

$u = c_1 z + c_2 z^2 + \dots + c_4 z^4 + \mathcal{O}(z_5)$

$c_1 \in R^*$ b/c it has to be a local parameter

similarly

in terms of u (at least up to degree 5)

$x = z^{-2} + \dots = c_1^2 u^{-2} + ? u^{-1} + ? u^0 + ? u^1 + \dots\dots$

$y = z^{-3} + \dots = c_1^3 u^{-3} + \dots$ no sense

But conditions (i), (ii) are not satisfied.

but locally, in the flat topology on R

$R \xrightarrow[\text{faithfully flat}]{} R[t]/t^6 = c_1$

locally we may therefore assume $c_1 = t^6$

replace $x \to c_1^2 x$, $y \to c_1^3 y$ then $z \to c_1^{-1} z$

$\Rightarrow z = u + \dots$

scaling, x looks like $u^{-2} + \dots$ $\Rightarrow$ (i)

y -- -- $u^{-3} + \dots$ $\Rightarrow$ (ii)

now $z = \frac{x}{y}$ doesn't satisfy (iii)

but know that

$z = u + c \cdot u^2 + \dots$ b/c (i)

change $y \to y + sx$

$\frac{x}{y} \to \frac{x}{y+sx} = z - sz^2 + o(z^3)$

$\exists!\, s$ $(s = c)$ s.t. the new z satisfies $z = u + d.u^3 + \dots$

replacing $x \to x + r$
$y \to z + r u^3$ $\qquad \exists!\, r$

$z = u + m u^4 + \dots$

and with $y \to y + z$

$z \to z - t z^4 + \dots$ $\quad \exists!\, t$ s.t. new z $= u + O(5)$

$\Rightarrow \exists!$ choices s.t. (iii) satisfied.

Oct. 1

<u>thm</u>:
$$\begin{array}{c} \mathcal{M}_{Weier} \\ \downarrow \\ \mathcal{M}_{FG} \end{array}$$
is representable

<u>Prop.</u>
$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "{\substack{y^2 + a_1xy + a_3y \\ = x^3 + a_2x^2 + a_4x + a_6}}"] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{M}_{Weier} \arrow[d] \\ \mathcal{M}^{(5)}_{FG} \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

formal gps + a parameter mod degree 5

where $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$

(it's the proposition proved last time)

<u>Pf</u> (thm)

$$\begin{tikzcd} \operatorname{Spec} A[b_5, b_6, \dots] \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[r] \arrow[d] & \mathcal{M}_{Weier} \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \mathcal{M}^{(5)}_{FG} \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

$L = \pi_0 MP = \mathbb{Z}[a_1, \dots]$

on $y + a_1 xy + \dots$ we have coordinates $z = \frac{x}{y}$

new coord.

Def. A map $\begin{array}{c}\mathcal{N}\\ \downarrow\\ \mathcal{M}\end{array}$ is $\begin{cases}\text{flat}\\ \text{étale}\\ \text{a cover}\end{cases}$ if it is representable

and, for any $\mathcal{U} \to \mathcal{M}$, $\begin{array}{ccc}\mathcal{W} & \longrightarrow & \mathcal{N}\\ \downarrow & \text{p.b.} & \downarrow\\ \mathcal{U} & \longrightarrow & \mathcal{M}\end{array}$ the map $\mathcal{W} \to \mathcal{U}$ is $\begin{cases}\text{flat}\\ \text{étale}\\ \text{a cover}\end{cases}$.

We get a Grothendieck topology $(\mathcal{M})_{et}$.

$p > 3$, $\mathcal{M}_{Weier} \otimes \mathbb{Z}/p$ reduce mod p

$$\begin{array}{c}\operatorname{Spec} \mathbb{Z}/p[a_6]\\ \Big\downarrow {\scriptstyle y^2 = x^3 + x + a_6}\\ \mathcal{M}_{Weier} \otimes \mathbb{Z}/p\end{array}$$

claim: this is étale.

Since $\mathcal{M}_{Weier} \otimes \mathbb{Z}/p = \text{ass}(\operatorname{Spec} \underbrace{\mathbb{Z}/p[c_4, c_6]}_{R_0}, R_0[\lambda^{\pm 1}])$

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}/p[a_6] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}/p[c_4, c_6] \arrow[r, "\text{is a cover}"'] & \mathcal{M}_{Weier} \otimes \mathbb{Z}/p
\end{tikzcd}$$

$c_4 \longmapsto \lambda^4 c_4$
$c_6 \longmapsto \lambda^6 c_6$

suffices to check that this is étale

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{Z}/p[a_6, \lambda^{\pm 1}] \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}/p[a_6] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}/p[c_4, c_6, \lambda^{\pm 1}] \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}/p[c_4, c_6] \arrow[d] \\
\operatorname{Spec} \mathbb{Z}/p[c_4, c_6] \arrow[r] & \mathcal{M}_{Weier} \otimes \mathbb{Z}/p
\end{tikzcd}$$

$$\begin{array}{cc} 0 & a_6 \\ \uparrow & \uparrow \\ c_4 & c_6 \end{array}$$

to work out what this map is:

so $\mathbb{Z}/p[c_4, c_6] \longrightarrow \mathbb{Z}/p[a_6, \lambda^{\pm 1}]$

$$c_6 \mapsto \lambda^6 a_6$$
$$c_4 \mapsto \lambda^4$$

$$\begin{tikzcd} \mathbb{Z}/p[c_4, c_6] \arrow[r] & \mathbb{Z}/p[a_6, \lambda^{\pm 1}] \\ \mathbb{Z}/p[c_4^{-1}, c^6] \arrow[u] \arrow[ur, "\text{add a 4th root of } c_4"'] & \end{tikzcd}$$

$\mathbb{Z}/p[a_6, \lambda^{\pm 1}] = \lambda^{-1}\, \mathbb{Z}/p[c_4, c_6][\lambda] / \lambda^4 = c_4$ flat and unramified.

<u>Thm</u>:

- $\exists$ a sheaf $\mathcal{O}^{\mathrm{Top}}$ on $(\mathcal{M}_{\mathrm{Weier}})_{\acute{e}t}$ with values in A_∞ ring spectra with the property that
$$\mathcal{O}(\mathrm{Spec} R \xrightarrow{\mathcal{E}} \mathcal{M}_{\mathrm{Weier}})$$
is an even periodic cohomology theory E ($\mathcal{E}$ = elliptic curve) with $\pi_0 E = R$ and FG = completion of $\mathcal{E}$.

<u>Rem</u>: $\pi_0 \mathcal{O}^{\mathrm{Top}}$ is a presheaf of rings and ass $\pi_0 \mathcal{O}^{\mathrm{Top}}$ a sheaf of functions. Similarly ass $\pi_{2n} \mathcal{O}^{\mathrm{Top}} = \omega^n$.

- Unique up to homotopy equivalence (of sheaves of A_∞ ring spectra)

Moreover $\mathcal{O}^{\mathrm{Top}}(\mathcal{M}_{\mathrm{Weier}})$ is E_∞.

<u>Def</u>: $tmf = (-1)$-connected cover of $\mathcal{O}^{\mathrm{Top}}(\mathcal{M}_{\mathrm{Weier}})$

(functorially: spectra $\rightsquigarrow$ elliptic curve)

About the notions of A_∞, E_∞.

DICTIONARY

Spectra	Abelian gps	
$\wedge$ smash product (derived from smash product spaces)	$\otimes$ Tensor product	
$\vee$ wedge product (coproduct of spectra)	$\oplus$ Whitney sum	
A_∞ ring spectrum	associative algebra R with $R\otimes R \to R$	(1)
E_∞ ring spectrum (symmetric algebra in cat of spectra)	commutative ring	(2)
$S^0 \wedge E \cong E$	$\mathbb{Z} \otimes A = A$ (unit)	
$X \wedge_R Y$	$M \otimes_R N$ (formally works the same)	
$S^0 \to R$	$\mathbb{Z} \xrightarrow[\text{unit}]{1} R$	

Notes:

(1) formulation of associativity $\rightsquigarrow$ diagrams involving triple $\otimes$
$(A\otimes B)\otimes C \neq A \otimes (B\otimes C)$ but canonically isomorphic.
this becomes a deeper issue in topology. Was once a large problem (A_∞: A for Associative, ∞ for infinitely many conditions). But now we have a different way of setting up smash product ($\wedge$ strictly associative)