Write

$$\langle E\rangle \vee \langle E'\rangle = \langle E \vee E'\rangle$$

$$\langle E\rangle \wedge \langle E'\rangle = \langle E \wedge E'\rangle$$

This makes the set of Bousfield classes of spectra a distributive lattice. If $\Sigma^k E \xrightarrow{v} E$ is a map, we write

$$E/v = \text{cofiber}\,(\Sigma^k E \xrightarrow{v} E)$$

$$v^{-1}E = \text{colim}\,(E \xrightarrow{v} \Sigma^{-k}E \xrightarrow{v} \Sigma^{-2k}E \to \cdots)$$

<u>Lemma</u> (Ravenel) $\langle E\rangle = \langle E/v\rangle \vee \langle v^{-1}E\rangle$.

pf Must show that E and $E/v \vee v^{-1}E$ have the same acyclic spectra. Suppose $(E/v)_* C = 0$ and $(v^{-1}E)_* C = 0$. Then

$$\cdots \to E_* C \xrightarrow[\sim]{v} E_* C \to \underset{\substack{\| \\ 0}}{(E/v)_* C} \to \cdots$$

$$E_* C \xrightarrow{\sim} (v^{-1}E)_* C = 0.$$

Similar in the opposite direction. //

MP , $\pi_0 MP$ = Lazard ring

Spec $\pi_0 MP$ = moduli space of formal grps.

$v_0 = p \in \pi_0 MP$

$v_1 \in \pi_0(MP/_p) = (\pi_0 MP)/_p$

$v_1^{-1} MP/_p \leftrightarrow$ formal grps of ht 1.

$MP/_{(p,v_1)} \leftrightarrow$ formal grps. of ht. > 1.

$v_2 \in \pi_0(MP/_{(p,v_1)}) = (\pi_0 MP)/_{(p,v_1)}$

$v_2^{-1} MP/_{(p,v_1)}$

$MP/_{(p,v_1,v_2)}$

$\vdots$

Thm (Ravenel, Johnson, Wilson)

$$\langle v_n^{-1} MP/_{(p,v_1,\ldots,v_{n-1})} \rangle = \langle K(n) \rangle .$$

//

Trying to make the sheaf (of E_∞-ring spectra) $\mathcal{O}^{top}$ on the moduli stacks $\mathcal{M}_{Weier}$ and $\mathcal{M}_{Ell}$.

a) We will do this one prime p at a time.

b) At each p further break the problem into super-singular and ordinary parts.

In more detail,

a) Arithmetic square (Sullivan).

$$M_p = S^0 \cup_p e^1; \quad H_i M_p = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i = 0, \\ 0 & \text{else}. \end{cases}$$

Define the p-completion of X to be the Bousfield localization w.r.t. M_p:

$$X_p := L_{M_p} X .$$

The rationalization $X_{\mathbb{Q}}$ is the loc. w.r.t. the Eilenberg-MacLane spectrum $H\mathbb{Q}$.

<u>Lemma</u> The following square is homotopy cartesian.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_p \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & (\prod_p X_p)_{\mathbb{Q}} \end{tikzcd}$$ //

b) Hasse square.

$K(n)$ = nth Morava K-theory

$K(n)_*(pt) = \mathbb{F}_p[u^{\pm 1}]$, $\deg u = 2$.

Prop The following sq. is htpy. cartesian.

$$\begin{tikzcd} L_{K(1)\vee K(2)} X \arrow[r] \arrow[d] & L_{K(2)} X \arrow[d] \\ L_{K(1)} X \arrow[r] & L_{K(1)} L_{K(2)} X \end{tikzcd}$$

($\sim$ supersing.)

(ordinary $\sim$)

In addition, if $X = L_{K(0)\vee K(1)\vee K(2)} X$, then $L_{K(1)\vee K(2)} X = X_p$. Here $K(0)_*(p) = \mathbb{Q}[u^{\pm 1}]$. //

pf Define Y to be the htpy. pull-back

$$\begin{tikzcd}
X \arrow[dr, dashed, "f"] \arrow[rr, bend left] \arrow[ddr, bend right] & & \\
& Y \arrow[r] \arrow[d] & L_{K(2)}X \arrow[d] \\
& L_{K(1)}X \arrow[r] & L_{K(1)}L_{K(2)}X
\end{tikzcd}$$

Must show that (i) $K(1)_*f$ and $K(2)_*f$ are iso., and (ii) Y is $K(1)\vee K(2)$-local.

$$\begin{tikzcd}
K(1)_*X \arrow[dr, dashed, "\sim"] \arrow[rr, bend left] \arrow[ddr, bend right, "\sim"'] & & \\
& K(1)_*Y \arrow[r] \arrow[d, "\sim"] \arrow[dr, phantom, "\Longleftarrow"] & K(1)_*L_{K(2)}X \arrow[d, "\sim"] \\
& K(1)_*L_{K(1)}X \arrow[r] & K(1)_*L_{K(1)}L_{K(2)}X
\end{tikzcd}$$

$$\begin{tikzcd}
K(2)_*X \arrow[dr, dashed, "\sim"] \arrow[rr, bend left, "\sim"] \arrow[ddr, bend right] & & \\
& K(2)_*Y \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & K(2)_*L_{K(2)}X \arrow[d] \\
& K(2)_*L_{K(1)}X \arrow[r, "\simeq"] & K(2)_*L_{K(1)}L_{K(2)}X
\end{tikzcd}$$

$$K(2)_*L_{K(1)}X = 0 \qquad K(2)_*L_{K(1)}L_{K(2)}X = 0$$

Have used $K(2)_* L_{K(1)} X = 0$. This is not a formal result. Will give reference later. This proves (i). For (ii), we must show that

$$K(1)_* Z = K(2)_* Z = 0 \Rightarrow [Z, Y]_* = 0.$$

This is immediate from definitions. //

<u>Note</u> The same proof works for the arithmetic square: The non-formal fact here is that $M_{p*} X_{\mathbb{Q}} = 0$. //

MP = periodic complex cobordism.

$\pi_0 MP$ = Lazard ring.

$MP \xrightarrow{p} MP \longrightarrow MP/(p)$

$[p](x) = u_1 x^p + \cdots \pmod p$; $u_1 \in \pi_0 MP/(p)$.

<u>Theorem</u> (Ravenel et. al.)

$$\langle u_1^{-1} MP/(p) \rangle = \langle K(1) \rangle$$

$$\langle u_2^{-1} MP/(p, u_1) \rangle = \langle K(2) \rangle .$$

//

We might see the proof of this later.

Consequence: Suppose E = even periodic mult. coh. theory. Choose a coordinate on the formal grp. G_E of E gives a map $MP \to E$. Then $u_1^{-1}E/p$ is a module over $u_1^{-1}MP/p$ and therefore local w.r.t. $u_1^{-1}MP/p$ and hence w.r.t. $K(1)$.

<u>Cor</u> $$L_{K(1)}E = \underset{\underset{k}{\longleftarrow}}{\mathrm{holim}}\, u_1^{-1}E/(p^k).$$

<u>Cor</u> $$L_{K(2)}E = \underset{\underset{a,b}{\longleftarrow}}{\mathrm{holim}}\, u_2^{-1}E/(p^a, u_1^b).$$

<u>ex</u> Let J be an elliptic curve over $R = \pi_0 E$ (torsion free) together with an iso. of G_E and the formal completion of J. Then

$$\pi_0\, \underset{\underset{k}{\longleftarrow}}{\mathrm{holim}}\, u_1^{-1}E/p^k = \underset{\underset{k}{\longleftarrow}}{\lim}\, \pi_0 u_1^{-1}E/p^k = R^{ord}$$

Similar

$$\pi_0\, \underset{\underset{a,b}{\longleftarrow}}{\mathrm{holim}}\, u_2^{-1}E/(p^a, u_1^b) \longleftrightarrow \text{completion at supersing. locus.}$$

//

$$\begin{tikzcd} \mathcal{O}_p^{\text{top}} \arrow[r] \arrow[d] & \mathcal{M}^{ss} \arrow[d] \\ \mathcal{M}^{\text{ord}} \arrow[r] & \eta \end{tikzcd}$$

~ everything here is governed by formal group (Serre–Tate)

here in K(1)-local htpy th.
— can calc. everything.

Deformation of formal groups :

k = perfect field of char. $p > 0$

Γ = formal grp. of ht. n over k.

B = complete local ring

$k \xrightarrow{t} B/m$ $\quad m \subset B$ max'l ideal.

A __deformation__ of Γ to B is a formal group G over B together with an iso. $G/m \xleftarrow{\sim} t^*\Gamma$ of formal grps over B/m. Let

$$\mathrm{Def}_\Gamma(B, k \xrightarrow{t} B/m) = \{\text{deformations of } \Gamma\}.$$

Thm (Lubin-Tate) There exists a formal group G_u over $W(k)[[u_1, \ldots, u_{n-1}]]$ with the property that $(u_n = 1)$

$$[p]_{G_u}(x) \equiv u_k x^{p^k} + \cdots \quad \text{mod } (p, u_1, \ldots, u_{k-1}).$$

Moreover, every such G_u is a universal def:

$$\mathrm{Hom}(W(k)[[u_1, \ldots, u_{n-1}]], B) \xrightarrow{\sim} \mathrm{Def}_{\Gamma}(B).$$

$$f \longmapsto f^* G_u$$

//

Def Let E be an even periodic mult. coh. th. with $\pi_0 E$ commutative. Say that E is a universal deformation if

(i) $\pi_0 E$ = compl. local ring w/ max. ideal $\mathfrak{m}$.

(ii) G_E is a universal def. of $G_E/\mathfrak{m}$.

Theorem The functor

$$\left\{ \begin{array}{l} E_\infty \text{ universal} \\ \text{def. } E \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{Formal grps. of} \\ \text{ht.} < \infty \text{ over perfect} \\ \text{fields of char. } p > 0 \end{array} \right\}$$

is an equivalence of categories. (The mapping spaces on LHS are htpy discrete.) //

Thm There is a 'weak' equivalence of categories

$$E_\infty\text{-universal deformations} \longrightarrow \text{Formal groups over perfect fields}$$

Two steps:

- constructing enough E_∞-ring spectra.
- analyzing the space of maps between them

As for any moduli problem, the two steps are related. Begin with 2nd step.

Universal coeff. thm:

E ring spectrum, E_* commutative. Relate $E^*(X)$ to $E_*(X)$. Canonical map

$$E^*(X) \longrightarrow \mathrm{Hom}_{E_*}(E_*(X), E_*)$$

$$X \xrightarrow{f} E \longmapsto \pi_*(E\wedge X) \xrightarrow{E\wedge f} \pi_*(E\wedge E) \xrightarrow{\mu} \pi_* E$$

To get universal coeff. formula,

1) Find enough spectra V for which

$$E^*(V) \xrightarrow{\sim} \mathrm{Hom}_{E_*}(E_*(V), E_*)$$

Enough: Given X, $\exists V \to X$ s.t. $E_*(V) \twoheadrightarrow E_*(X)$.

2) Make a resolution of X by such V:

$$\cdots \to V_2 \to V_1 \to V_0 \to X$$

$$\cdots \to E_*(V_2) \to E_*(V_1) \to E_*(V_0) \to E_*(X) \quad \text{exact}.$$

Get spectral sequence

$$E_1 = \bigoplus E^*(V_n) \Longrightarrow E^*(X)$$

Since $E^*(V_n) \xrightarrow{\sim} \mathrm{Hom}_{E_*}(E_*(V_n), E_*)$, get

$$E_2 = \mathrm{Ext}_{E_*}(E_*(X), E_*) \Rightarrow E^*(X).$$

Discuss the construction of the sp. seq. in more detail. In (2) we really need to use a simplicial resolution

$$\cdots \; V_2 \;\substack{\textstyle\rightarrow\\[-2pt]\textstyle\leftarrow\\[-2pt]\textstyle\rightarrow\\[-2pt]\textstyle\leftarrow\\[-2pt]\textstyle\rightarrow}\; V_1 \;\substack{\textstyle\rightarrow\\[-2pt]\textstyle\leftarrow\\[-2pt]\textstyle\rightarrow}\; V_0 \to X.$$

This does not require that maps be additive, so works also in the situation of ring spectra, where we cannot add maps.

Let E, F be A_∞-ring spectra. Wish to understand

$$\pi_* A_\infty(E, F).$$

As in the universal coeff. thm., the first approx. is given by the map

$$\pi_* A_\infty(E, F) \longrightarrow (F_*\text{-alg})(F_*E, F_*)$$

$$E \xrightarrow{f} F \longmapsto F_*E \xrightarrow{f_*} F_*F \xrightarrow{\mu} F_*$$

An elem. $\alpha \in \pi_n(A_\infty(E,F), x_0)$ corresponds to a htpy class of lifts

$$\begin{tikzcd}
 & F^{S^n} \arrow[d, "ev"] \\
E \arrow[r, "x_0"'] \arrow[ur, dashed, "\alpha"] & F
\end{tikzcd}$$

Algebraic approximation

$$\begin{tikzcd}
 & F_*[\varepsilon]/(\varepsilon^2) \arrow[d] \\
F_*E \arrow[r, "\alpha_0"'] \arrow[ur, dashed] & F_*
\end{tikzcd}$$

$y \longmapsto \alpha_0(y) + D(y)\varepsilon$ $\qquad$ $\deg \varepsilon = n$

D – a derivation.

So alg. approx. is

$$\pi_n(A_\infty(E,F), x_0)$$

$$\longrightarrow \mathrm{Der}_{F_*}(F_*E, x_0^* F_*).$$

Look for good V

- $F_*(V)$ projective
- $F^*(V) \xrightarrow{\sim} \mathrm{Hom}_{F_*}(F_*(V), F_*)$
- $F_*(V) \otimes_{F_*} F_*(X) \xrightarrow{\sim} F_*(V \wedge X)$.

Suppose V is good. Let

$T(V)$ = free A_∞-ring spectrum gen. by V

$$T(V) \sim \bigvee_{n=0}^{\infty} V^{\wedge n}$$

$$F_*(V^{\wedge n}) \xleftarrow{\sim} \overbrace{F_*(V) \otimes_{F_*} \cdots \otimes_{F_*} F_*(V)}^{n} \quad \text{so}$$

$$F_*(T(V)) \xleftarrow{\sim} T_{F_*}(F_*(V)) \quad \text{tensor alg.}$$

Next, construct enough good Y (can be done for all Landweber exact coh. th.):

Given E, construct a simpl. res.

$$P_\cdot \longrightarrow E$$

s.t. each $P_n = T(V_n)$, V_n good, s.t. $F_* P_\cdot \to F_* E$ is a resolution.

Since $P_\cdot$ is a simpl. A_∞-ring sp., $A_\infty(P_\cdot, F)$ is a cosimpl. space with $\mathrm{Tot}\, A_\infty(P_\cdot, F) \xleftarrow{\sim} A_\infty(E, F)$. Consider htpy. spectral seq. for this cosimpl. space. Since $P_n = T(V_n)$,

$$A_\infty(P_n, F) \simeq \mathrm{Spectra}(V_n, F),$$

so

$$\pi_* A_\infty(P_n, F) \simeq F^*(Y_n)$$

$$\xrightarrow{\sim} \mathrm{Hom}_{F_*}(F_*(Y_n), F_*).$$

This is not a natural identification, though. The natural identification is:

$$\pi_0 A_\infty(P_n, F) = (F_*\text{-alg})(F_* P_n, F_*)$$

$$\pi_k(A_\infty(P_n, F), x_0) = \mathrm{Der}_{F_*}(F_* P_n, x_0^* F_{*+k}) \quad k>0.$$

led to the following alg. recipe:

1) Choose an F_*-alg. hom. $F_* E \xrightarrow{\alpha} F_*$

2) Form a simpl. res. of $F_* E$ by free assoc. F_*-alg. $R_\cdot \xrightarrow{\sim} F_* E$.

Form the cochain complex

$$\mathrm{Der}_{F_*}(R_\cdot, \alpha^* F_*)$$

Cohomology of this cx. are (by def.) the higher derived functors of Der:

$$\mathrm{Der}^s_{F_*}(F_* E, \alpha^* F_*). \qquad \text{(Quillen)}$$

So get a spectral sequence

$$\mathrm{Der}^{-s}_{F_*}(F_* E, F_{*+t}) \Rightarrow \pi_{s+t} A_\infty(E, F).$$

Need to discuss base-points.