

Write

$$\langle E \rangle \vee \langle E' \rangle = \langle E \vee E' \rangle$$

$$\langle E \rangle \wedge \langle E' \rangle = \langle E \wedge E' \rangle$$

This makes the set of Bousfield classes of spectra a distributive lattice. If $\Sigma^k E \xrightarrow{\nu} E$ is a map, we write

$$E/\nu = \text{cofiber}(\Sigma^k E \xrightarrow{\nu} E)$$

$$\nu^{-1} E = \text{cdim}(E \xrightarrow{\nu} \Sigma^{-k} E \xrightarrow{\nu} \Sigma^{-2k} E \rightarrow \dots)$$

Lemma (Ravenel) $\langle E \rangle = \langle E/\nu \rangle \vee \langle \nu^{-1} E \rangle$.

pf: Must show that E and $E/\nu \vee \nu^{-1} E$ have the same acyclic spectra. Suppose $(E/\nu)_* C = 0$ and $(\nu^{-1} E)_* C = 0$. Then

$$\dots \rightarrow E_* C \xrightarrow{\sim} E_* C \rightarrow (E/\nu)_* C \rightarrow \dots$$

||
0

$$E_* C \xrightarrow{\sim} (\nu^{-1} E)_* C = 0$$

Similar in the opposite direction. //

MP, $\pi_0 MP =$ Lazard ring

Spec $\pi_0 MP =$ moduli space of formal grps.

$v_0 = p \in \pi_0 MP$

$v_1 \in \pi_0 (MP/p) = (\pi_0 MP)/p$

$v_1^{-1} MP/p \leftarrow$ formal grps of ht 1.

$MP/(p, v_1) \leftarrow$ formal grps. of ht. > 1 .

$v_2 \in \pi_0 (MP/(p, v_1)) = (\pi_0 MP)/(p, v_1)$

$v_2^{-1} MP/(p, v_1)$

$MP/(p, v_1, v_2)$

;

Thm (Ravenel, Johnson, Wilson)

$\langle v_n^{-1} MP / (p, v_1, \dots, v_{n-1}) \rangle = \langle K(n) \rangle$ //

Trying to make the sheaf (of E_∞ -ring spectra) \mathcal{O}^{top} on the moduli stacks M_{Weier} and M_{Ell} .

a) We will do this one prime p at a time.

b) At each p further break the problem into super-singular and ordinary parts.

In more detail,

a) Arithmetic square (Sullivan).

$$M_p = S^0 \cup_p e^1; \quad H_* M_p = \begin{cases} \mathbb{Z}/p\mathbb{Z} & i=0, \\ 0 & \text{else.} \end{cases}$$

Define the p -completion of X to be the Bousfield localization w.r.t. M_p :

$$X_p := L_{M_p} X.$$

The rationalization $X_{\mathbb{Q}}$ is the loc. w.r.t. the Eilenberg-MacLane spectrum $H\mathbb{Q}$.

Lemma The following square is homotopy cartesian.

$$\begin{array}{ccc}
 X & \longrightarrow & \prod_P X_P \\
 \downarrow & & \downarrow \\
 X_Q & \longrightarrow & \left(\prod_P X_P \right)_Q \quad //
 \end{array}$$

b) Hasse square.

$K(n)$ = n th Morava K -theory

$$K(n)_*(p) = \mathbb{F}_p[u^{\pm 1}], \quad \deg u = 2.$$

Prop The following sq. is htpy. cartesian.

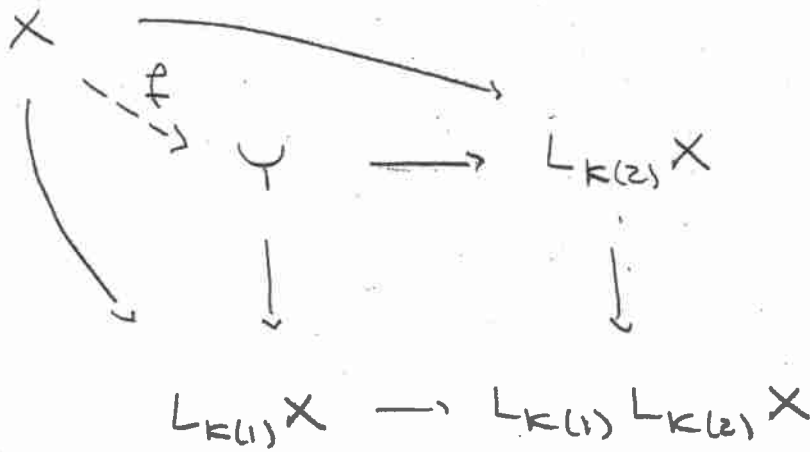
$$\begin{array}{ccc}
 L_{K(1) \vee K(2)} X & \longrightarrow & L_{K(2)} X \quad (\sim \text{supersing.}) \\
 \downarrow & & \downarrow
 \end{array}$$

diag. \sim) $L_{K(1)} X \longrightarrow L_{K(1)} L_{K(2)} X$

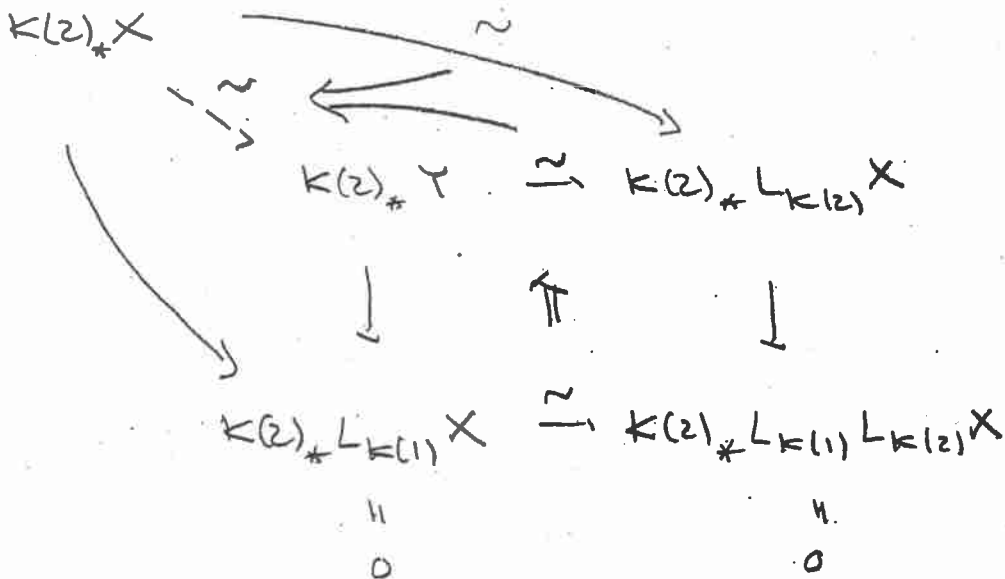
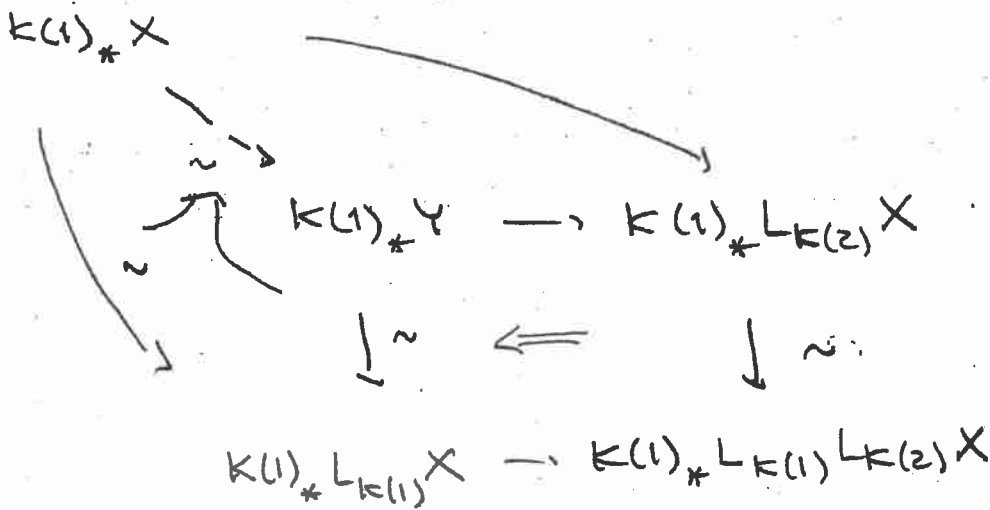
In addition, $\mathbb{P}^1 X = L_{K(0) \vee K(1) \vee K(2)} X$.

then $L_{K(1) \vee K(2)} X = X_p$. Here $K(0)_*(p) = \mathbb{Q}[u^{\pm 1}]$. //

Pf Define Y to be the htpy. pull-back



Must show that (i) $K(1)_* f$ and $K(2)_* f$ are iso., and (ii) Y is $K(1) \vee K(2)$ -local.



Have used $K(2)_* L_{K(1)} X = 0$. This is not a formal result. Will give reference later. This proves (i). For (ii), we must show that

$$K(1)_* Z = K(2)_* Z = 0 \Rightarrow [Z, Y]_* = 0.$$

This is immediate from definitions. "

Note The same proof works for the arithmetic square. The non-formal fact here is that $M_p * X_Q = 0$. "

MP = periodic complex cobordism.

$\pi_0 MP$ = Lazard ring.

$$MP \xrightarrow{p} MP \rightarrow MP/(p)$$

$$[p](x) = u_1 x^p + \dots \pmod{p} \quad ; \quad u_1 \in \pi_0 MP/(p).$$

Theorem (Ravenel et. al.)

$$\langle u_1^{-1} MP/(p) \rangle = \langle K(1) \rangle$$

$$\langle u_2^{-1} MP/(p, u_1) \rangle = \langle K(2) \rangle$$

We might see the proof of this later.

Consequence: Suppose E = even periodic mult. coh. theory. Choose a coordinate on the formal grp. G_E of E gives a map $MP \rightarrow E$. Then $u_1^{-1}E/p$ is a module over $u_1^{-1}MP/p$ and therefore local w.r.t. $u_1^{-1}MP/p$ and hence w.r.t. $K(1)$.

$$\underline{\text{Cor}} \quad L_{K(1)} E = \varprojlim_k u_1^{-1} E / (p^k)$$

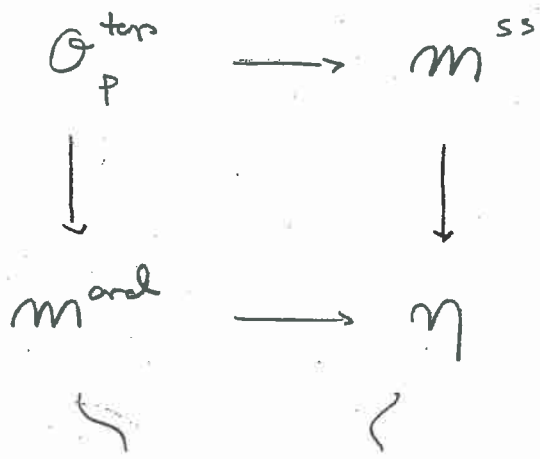
$$\underline{\text{Cor}} \quad L_{K(2)} E = \varprojlim_{a,b} u_2^{-1} E / (p^a, u_1^b)$$

ex let J be an elliptic curve over $R = \pi_0 E$ (torsion free) together with an isom. of G_E and the formal completion of J . Then

$$\pi_0 \varprojlim_k u_1^{-1} E / p^k = \varprojlim_k \pi_0 u_1^{-1} E / p^k = R^{\text{ord}}$$

Similar

$$\pi_0 \varprojlim_{a,b} u_2^{-1} E / (p^a, u_1^b) \longleftrightarrow \text{completion at supersing. locus.}$$



everything here is governed by formal group (Serre-Tate)

here in K(1)-local httpz th. — can calc. everything.

Deformation of formal groups :

k = perfect field of char. $p > 0$

Γ = formal grp. of ht. n over k .

B = complete local ring

$k \xrightarrow{t} B/m$ $m \subset B$ max'l ideal.

A deformation of Γ to B is a formal group G over B together with an iso. $G/m \xrightarrow{\sim} t^* \Gamma$ of formal grps over B/m . Lit.

$$\text{Def}_{\Gamma}(B, k \xrightarrow{t} B/m) = \{ \text{deformations of } \Gamma \}.$$

Thm (Lubin-Tate) There exists a formal group G_u over $W(k)[[u_1, \dots, u_{n-1}]]$ with the property that $(u_n = 1)$

$$[p]_{G_u}(x) \equiv u_k x^k + \dots \pmod{(p, u_1, \dots, u_{k-1})}$$

Moreover, every such G_u is a universal def:

$$\text{Hom}(W(k)[[u_1, \dots, u_{n-1}], B) \xrightarrow{\sim} \text{Def}_p(B)$$

$$f \longmapsto f^* G_u \quad //$$

Def Let E be an even periodic mult. coh. th. with $\pi_0 E$ commutative. Say that E is a universal deformation if

(i) $\pi_0 E = \text{compl. local ring w/ max. ideal } \mathfrak{m}$.

(ii) G_E is a universal def. of G_E / \mathfrak{m} .

Theorem The functor

$$\left\{ \begin{array}{l} E_{\infty} \text{ universal} \\ \text{def. } E \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Formal grps. of} \\ \text{ht. } < \infty \text{ over perfect} \\ \text{fields of char. } p > 0 \end{array} \right\}$$

is an equivalence of categories. (The mapping spaces on LHS are ht_p discrete.) //

Thm There is a 'weak' equivalence of categories

$$E_{\infty}\text{-universal deformations} \longrightarrow \text{Formal groups over perfect fields}$$

Two steps:

- constructing enough E_{∞} -ring spectra.
- analyzing the space of maps between them

As for any moduli problem, the two steps are related. Begin with 2nd step.

Universal coeff. thm:

E ring spectrum, E_* commutative. Relate $E^*(X)$ to $E_*(X)$. Canonical map

$$E^*(X) \cong \text{Hom}_{E_*}(E_*(X), E_*)$$

$$X \xrightarrow{p} E \xrightarrow{\quad} \pi_* (E \wedge X) \xrightarrow{E \wedge p} \pi_* (E \wedge E) \xrightarrow{\sim} \pi_* E$$

To get universal coeff. formula,

- 1) Find enough spectra V for which

$$E^*(V) \xrightarrow{\sim} \text{Hom}_{E_*}(E_*(V), E_*)$$

Enough: Given X , $\exists V \rightarrow X$ s.t. $E_*(V) \rightarrow E_*(X)$.

2) Make a resolution of X by such V :

$$\cdots \rightarrow V_2 \rightarrow V_1 \rightarrow V_0 \rightarrow X$$

$$\cdots \rightarrow E_*(V_2) \rightarrow E_*(V_1) \rightarrow E_*(V_0) \rightarrow E_*(X) \quad \text{exact}$$

Get spectral sequence

$$E_1 = \bigoplus E^*(V_n) \Rightarrow E^*(X)$$

Since $E^*(V_n) \xrightarrow{\sim} \text{Hom}_{E_*}(E_*(V_n), E_*)$, get

$$E_2 = \text{Ext}_{E_*}(E_*(X), E_*) \Rightarrow E^*(X).$$

Discuss the construction of the sp. seq. in more detail. In (2) we really need to use a simplicial resolution

$$\cdots V_2 \begin{matrix} \rightrightarrows \\ \rightleftarrows \\ \rightleftarrows \\ \rightrightarrows \end{matrix} V_1 \begin{matrix} \rightrightarrows \\ \rightleftarrows \\ \rightleftarrows \\ \rightrightarrows \end{matrix} V_0 \rightarrow X$$

This does not require that maps be additive, so works also in the situation of ring spectra, where we cannot add maps.

Let E, F be A_∞ -ring spectra. Wish to understand

$$\pi_* A_\infty(E, F).$$

As in the universal coeff. thm., the first approx. is given by the map

$$\pi_* A_\infty(E, F) \rightarrow (F_*\text{-alg})(F_*E, F_*)$$

$$E \xrightarrow{f} F \mapsto F_*E \xrightarrow{f_*} F_*F \xrightarrow{\mu} F_*$$

An elem. $\alpha \in \pi_n(A_\infty(E, F), \kappa_0)$ corresponds to a homy class of lifts

$$\begin{array}{ccc} & & F S^n \\ & \nearrow \alpha & \downarrow \kappa \\ E & \xrightarrow{\kappa_0} & F \end{array}$$

Algebraic approximation

$$\begin{array}{ccc} & \nearrow \kappa_0(y) + D(y) \in & F_*[E]/(E^2) \\ & \nearrow \eta & \downarrow \\ F_*E & \xrightarrow{\kappa_0} & F_* \end{array} \quad \text{deg } E = n$$

D - a derivation.

So alg. approx. is:

$$\pi_n(A_\infty(E, F), x_0)$$

$$\rightarrow \text{Der}_{F_*}(F_*E, x_0^* F_*)$$

Look for good V

- $F_*(V)$ projective
- $F^*(V) \xrightarrow{\sim} \text{Hom}_{F_*}(F_*(V), F_*)$
- $F_*(V) \otimes_{F_*} F_*(X) \xrightarrow{\sim} F_*(V \wedge X)$

Suppose V is good. Let

$T(V)$ = free A_∞ -ring spectrum
gen. by V

$$T(V) \sim \bigvee_{n=0}^{\infty} V^{\wedge n}$$

$$F_*(V^{\wedge n}) \xrightarrow{\sim} \underbrace{F_*(V) \otimes_{F_*} \cdots \otimes_{F_*} F_*(V)}_n \rightarrow$$

$$F_*(T(V)) \xrightarrow{\sim} T_{F_*}(F_*(V)) \quad \text{tensor alg.}$$

Next, construct enough good V (can be done for all Landweber exact coh. th.):

Given E , construct a simpl. res.

$$P_{\bullet} \longrightarrow E$$

s.t. each $P_n = T(V_n)$, V_n good, s.t. $F_* P_{\bullet} \rightarrow F_* E$ is a resolution.

Since P_{\bullet} is a simpl. A_{∞} -ring sp., $A_{\infty}(P_{\bullet}, F)$ is a cosimpl. space with $\text{Tot } A_{\infty}(P_{\bullet}, F) \xrightarrow{\sim} A_{\infty}(E, F)$. Consider htop. spectral seq. for this cosimpl. space. Since $P_n = T(V_n)$,

$$A_{\infty}(P_n, F) \simeq \text{Spectra}(V_n, F),$$

so

$$\pi_* A_{\infty}(P_n, F) \simeq F^*(V_n)$$

$$\xrightarrow{\sim} \text{Hom}_{F_*}(F_*(V_n), F_*).$$

This is not a natural identification, though. The natural identification is:

$$\pi_0 A_\infty(P_n, F) = (F_*\text{-alg}) (F_*P_n, F_*)$$

$$\pi_k(A_\infty(P_n, F), x_0) = \text{Der}_{F_*} (F_*P_n, x_0^* F_{*+k}), \quad k > 0.$$

led to the following alg. recipe:

1) Choose an F_* -alg. hom. $F_*E \xrightarrow{\alpha} F_*$

2) Form a simpl. res. of F_*E by free assoc. F_* -alg. $R \xrightarrow{\sim} F_*E$.

Form the cochain complex

$$\text{Der}_{F_*} (R, \alpha^* F_*)$$

Cohomology of this α are (by def.) the higher derived functors of Der :

$$\text{Der}_{F_*}^s (F_*E, \alpha^* F_*). \quad (\text{Quillen})$$

So get a spectral sequence

$$\text{Der}_{F_*}^{-s} (F_*E, F_{*+t}) \Rightarrow \pi_{s+t} A_\infty(E, F).$$

Need to discuss base-points.