$\mathcal{E}$ = cat of bicomplexes of R-modules $X_{\bullet\bullet}$

internal or "space" direction

$$\begin{tikzcd}
\arrow[d] & \arrow[d] & \\
X_{01} \arrow[d] & X_{11} \arrow[l] \arrow[d] & \arrow[l] \\
X_{00} & X_{10} \arrow[l] & \arrow[l]
\end{tikzcd}$$

resolution direction

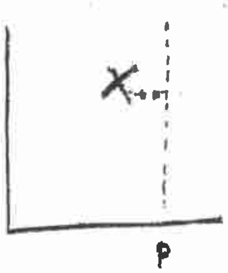
can't distinguish between 2 directions (internal one and resolution direction) but need to choose b/c we want analogies with $\pi_* P_\bullet \longrightarrow \pi_0 E$

$X_{\bullet\bullet} \longrightarrow Y_{\bullet\bullet}$ w. eq.

(take $\pi_*^{\text{vert}} X_\bullet$ still have the horiz. differentials)

$$\underbrace{\pi_*^{h}\, \pi_*^{\text{vert}} X_{\bullet\bullet}}_{E^2\text{-Term of the spectral sequence}} \xrightarrow{\sim} \pi_*^{h}\, \pi_*^{v}\, Y$$

Go back to the exact couple for it.



$$\begin{tikzcd}
F_{p-1} X \arrow[r] & F_p X \arrow[dl] \\
X_{p,*} &
\end{tikzcd}$$

$\longleftarrow$ SES of chain complexes

$\Big\Downarrow$

LES of htpy gps

$$\begin{tikzcd}
\pi_* F_{p-1} \arrow[r] & \pi_* F_p \arrow[d] \\
& \pi_* X_{p,*} \arrow[ul, dotted]
\end{tikzcd}$$

$$(F_p X_{\bullet\bullet})_n = \bigoplus_{\substack{i+j=n \\ i \le p}} X_{ij}$$

then sum over p.

$$\begin{tikzcd}
D' \arrow[r, "i"] & D' \arrow[dl] \\
E' \arrow[u] &
\end{tikzcd}$$

exact couple.

Derived exact couple

$D^2 = i(D^1) \subset D^1$

$E^2 = \pi^h \pi^v X_{\cdot\cdot}$

$$\begin{tikzcd} D^2 \arrow[rr] & & D^2 \arrow[dl] \\ & E^2 \arrow[ul, dashed] & \end{tikzcd}$$

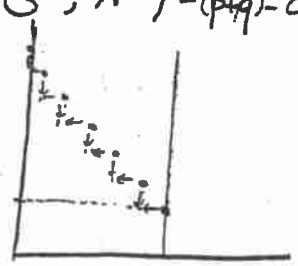
Rem: the E^2 weak eq. are also the maps $X \to Y$ for which

$$D^2X \xrightarrow{\simeq} D^2Y$$

$D^2X)_{p,q}$ = p+q cycles in F_pX / those which become boundaries in F_{p+1}

(all of these objects are birepresentable)

D. $S^{p,q}$ $\qquad C(S^{p,q}; X)$ = (p+q)-cycles in $F_p X$

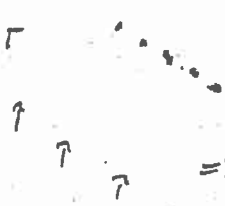
q  p

$\pi_p^h \pi_q^v S^{p,q} = R$

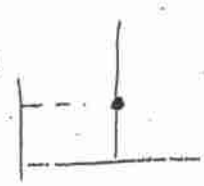
$\pi_*^h \pi_*^v S^{p,q} = 0 \qquad (*,*) \neq (p,q)$

copy of R for each dot

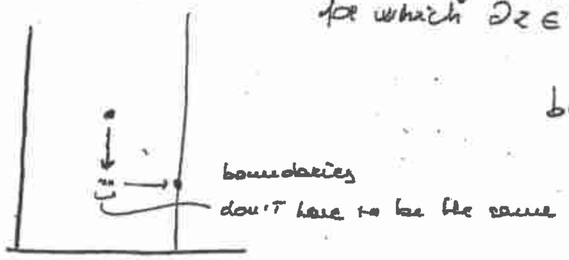
taking π^v ⇝ cancel



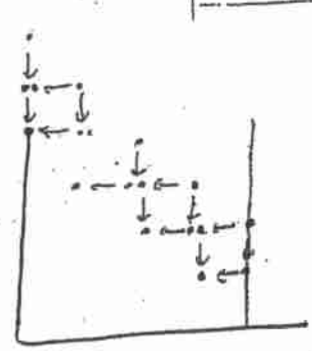
⇒ lasts only



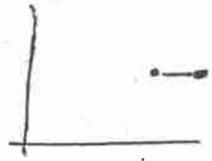
$D^{p+1,q}$ $\qquad C(D^{p+1,q}, X)$ = elts z of $F_{p+1}X$ of tot. deg. p+q+1 for which $\partial z \in F_p X$.



better



$D^{p+1,q}$

$\downarrow$ exact $\Rightarrow$ $\pi^v D^{p,q} =$  $\Rightarrow$ $\pi_x^h \pi_*^v D^{p,q} = 0$

$$C(D^{p+1,q}, X) = \{(x_0, \dots, x_{p+1}) \mid x_i \in X_{i,j},\ \partial^v x_{p+1} = 0,\ i+j = p+1\}$$

__Def__: $X_{\cdot\cdot} \to Y_{\cdot\cdot}$ is a fibration if

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ D^{p,q} \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd}$$

this is equivalent to saying that

$X_{p,q} \to X_{p,q}$ is surjective $(p,q) \neq (0,0)$

cycles in $X_{p,*} \twoheadrightarrow$ cycles $Y_{p,*}$, $p > 0$

again equivalent to

(surjective + surjective on cycles $\Leftrightarrow$)

$\pi_* X_{p,*} \to \pi_* Y_{p,*}$ epi for $p > 0$

__Thm__: this defines a model cat. structure on bi-complexes

$X \to Y$ is an acyclic fibration if

$$\begin{tikzcd} S^{p,q} \arrow[r] \arrow[d] & X \arrow[d] \\ D^{p+1,q} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ S^{p,q} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

ℓ: $\pi_{p,q}X - h_0 C(S^{p,q}, X)$

$$= C(S^{p,q}, X) / C(D^{p+1,q}, X)$$

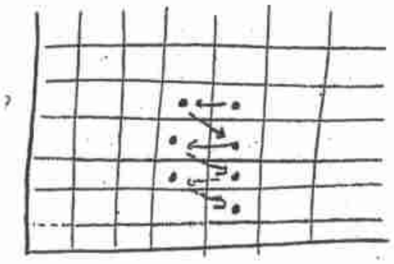
$= D^2_{p,q}(X)$ in the spectral sequence

bigraded homotopy groups

Exact couple gives the LES

$$\pi_{p-1,q+1}X \to \pi_{p,q}X \to \pi^h_p \pi^v_q X \to \pi_{p-1,q}X \to \cdots$$

SPIRAL
EXACT
SEQUENCE



p

e had some kind of resolution P_* of E.

$$P_0 \to E$$

$$\pi_k \pi_* P_\bullet = \begin{cases} 0 & , k > 0 \\ \pi_* E & , k = 0 \end{cases}$$

what does this condition mean in terms of the gps $\pi_{p,q}X$?

$\vdots$					
A_3	0	0			
A_2	0	0			
A_1	0	0			
A_0	0	0			

$\pi_p^h \pi_q^v X$

A_3				
A_2	A_3	A_4		
A_1	A_2	A_3		
A_0	A_1	A_2		

$\pi_{p,q} X$

tot deg $p+q$
in the p-th filtration

zero's here $\Rightarrow$ iso in A (maps $\searrow$)

by spiral exact sequence

i.e. groups constant on diagonals $\searrow$
and isomorphisms as maps

A reference for the non-linear obstruction theory that we will be getting into is: Goerss-Hopkins, Moduli Spaces of Commutative Ring Spectra. Discuss linear case further:

C = bi-complexes

model category structure with

fibrations: $X \to Y$ which are surjective and surjective on "vertical" cycles.

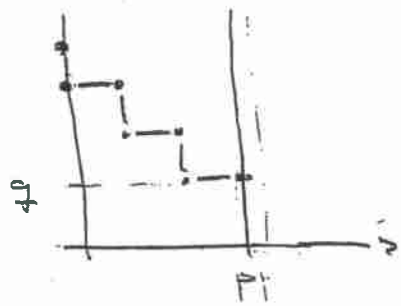
weak equiv.: iso. of $\pi_p^h \pi_q^v = E^2_{p,q}$, or equivalently, iso. of $\pi_{p,q} = D^2_{p,q}$

The two kinds of htpy. gps. are related by the spiral exact sequence

$$\cdots \to \pi_{p-1,q} \to \pi_{p,q} \to \pi_p^h \pi_q^v \to \pi_{p-2,q+1} \to \cdots$$

The functor $\pi_{p,q}$ will be co-represented in the homotopy category by the following bi-complex

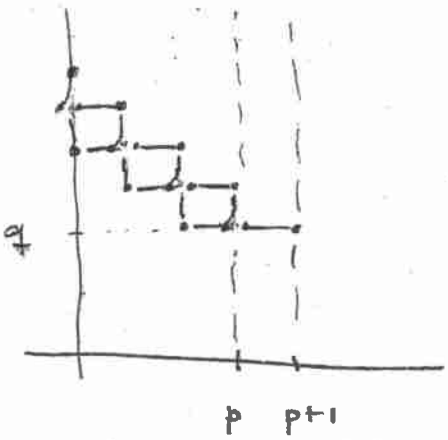
$S^{p,q}$:



$C(S^{p,q}, X)$ = cycles of deg. $p+q$ in F_pX.

Also introduced

$D^{p+1,q}$:



$C(D^{p+1,q}, X)$ = subset of $(F_{p+1}X)_{p+q+1}$ with

$$d^{tot} \in F_p X.$$

Terminology from model categories:

Y is <u>fibrant</u> if $Y \to \{0\}$ is a fibration.

X is <u>cofibrant</u> if $\{0\} \to X$ is a cofibration.

<u>ex</u> $S^{p,q}$ and $D^{p+1,q}$ are cofibrant.

To compute $\mathrm{Ho}\,\mathcal{C}(X,Y)$: factor

1)
$$\begin{tikzcd} Y \arrow[rr] \arrow[dr, hook, "\sim"'] & & \{0\} \\ & \tilde{Y} \arrow[ur, two heads] & \end{tikzcd}$$

2)
$$\begin{tikzcd} \{0\} \arrow[rr] \arrow[dr, hook] & & X \\ & \tilde{X} \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

3)
$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr, hook] & & \{0\} \\ & C\tilde{X} \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

Then there is a canonical iso.

$$\mathrm{Ho}\,\mathcal{C}(X,Y) \cong \mathcal{C}(\tilde{X},\tilde{Y}) / \mathcal{C}(C\tilde{X},\tilde{Y}).$$

<u>ex</u> $\mathrm{Ho}\,\mathcal{C}(S^{p,q},Y) \cong \pi_{p,q}Y$. //

In $\mathcal{C}$, the objects $D^{p,q}$ and $S^{p,q}$ are generators of the cofibrations and acyclic cofibrations, respectively. That means that

$$\begin{matrix} X \\ \downarrow \\ Y \end{matrix} \ \text{fibration} \iff \begin{tikzcd} \{0\} \arrow[r] \arrow[d, hook, "\sim"] & X \arrow[d] \\ D^{p,q} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array} \quad \begin{array}{c}\text{acyclic}\\ \text{fibration}\end{array} \quad \Longleftrightarrow \quad \begin{tikzcd} \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ S^{p,q} \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

We construct the fibrant replacement

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook] & & Y \\ & \tilde{Y} \arrow[ur, two heads, "\sim"'] & \end{tikzcd}$$

by Quillen's small object argument, which builds on Grothendieck's way of producing enough injectives. Let $X_0 = X$ and let X_1 be the push-out

$$\bigoplus_{\substack{S^{p,q} \to X \\ \downarrow \quad \downarrow \\ D^{p+1,q} \to Y}} S^{p,q} \longrightarrow \bigoplus D^{p+1,q}$$

$$\begin{tikzcd} \bigoplus S^{p,q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p-o.}"] & \bigoplus D^{p+1,q} \arrow[d] \arrow[ddr, bend left] & \\ X_0 \arrow[r] \arrow[drr, bend right] & X_1 \arrow[dr, dashed] & \\ & & Y \end{tikzcd}$$

and iterate

$$X_0 \to X_1 \to X_2 \to \cdots \to \varinjlim_n X_n = \tilde{Y} \to Y.$$

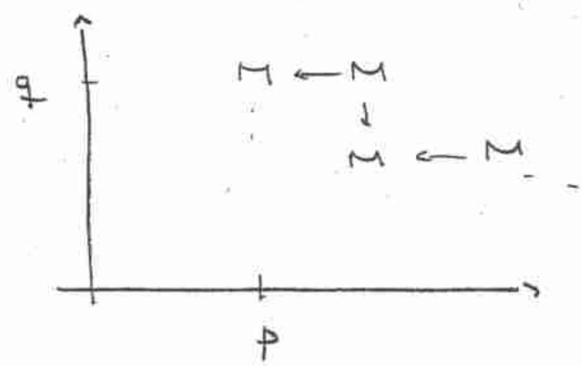
Eilenberg-MacLane objects:

M R-module

$$\pi_{s,t} K(M,p,q) = \begin{cases} M & (s,t) = (p,q), \\ 0 & \text{else}. \end{cases}$$

Construction I:

$K(M,p,q)$:



This is fibrant, so good for mapping into.

Construction II:

$$\pi_s \pi_t S^{p,q} = \begin{cases} R & (s,t) = (p,q), \\ 0 & \text{else}. \end{cases} \Rightarrow$$

$$\pi_{s,t} S^{p,q} = \begin{cases} R & s+t = p+q,\ s \geq p, \\ 0 & \text{else}. \end{cases}$$

Cone off the generator of $\pi_{p+1,q-1}$:

$$\begin{array}{ccc} S^{p+1,q-1} & \longrightarrow & D^{p+2,q-1} \\ \downarrow & \text{p-o.} & \downarrow \\ S^{p,q} & \longrightarrow & K(R,p,q) \end{array}$$

This give a cofibrant model of $K(R,p,q)$. More generally, If P is a projective R-module, we get a cofibrant Eilenberg-MacLane object

$$K(P,p,q) = P \otimes_R K(R,p,q).$$

For a general R-module M, first take a proj. presentation $P_1 \to P_0 \to M$ and form the p-o.

$$\begin{array}{ccc} S^{p,q} \otimes P_1 & \longrightarrow & D^{p+1,q} \otimes P_1 \\ \downarrow & \text{p-o.} & \downarrow \\ S^{p,q} \otimes P_0 & \longrightarrow & X_1 \end{array}$$

then

$$\pi_{s,t} X_1 = \begin{cases} M & (s,t) = (p,q) \\ 0 & s<p,\ s=p,\ t\neq q. \end{cases}$$

Next, cone off the higher htpy grps as when we construct Eilenberg-MacLane spaces in topology:

$$\begin{tikzcd} \bigoplus_{\pi_{p+1,*}X_1} S^{p+1,*} \arrow[r] \arrow[d] \arrow[dr, phantom, "p.o."] & \bigoplus_{\pi_{p+1,*}X_1} D^{p+2,*} \arrow[d] \\ X_1 \arrow[r] & X_2 \end{tikzcd}$$

and iterate to get

$$K(M,p,q) = \varinjlim_n X_n .$$

This is a cofibrant model. //

Suppose $M = P$ is projective. Using the two models of Eilenberg-MacLane objects, we get $(K(P,p,q) = P \otimes (S^{p,q} \cup D^{p+1,q-1}))$

$$\mathrm{Ho}\,\mathcal{C}(K(P,p,q), K(N,s,t))$$

$$= \begin{cases} \mathrm{Hom}(M,N) & (s,t) = (p,q) \text{ or } (p+2,q-1), \\ 0 & \text{else.} \end{cases}$$

or better

$$\mathrm{Ho}\,\mathcal{C}(K(P,p,q),\ K(N,s,t))$$

$$= \mathrm{Hom}\,(\pi_*^h \pi_*^v K(P,p,q),\ N_{s,t})\ .$$

More generally, I think,

$$\mathrm{Ho}\,\mathcal{C}(X,\ K(N,s,t)).$$

$$= \prod_k \mathrm{Ext}_R^k(\pi_{s-k}^h \pi_t^v X.,\ N).$$

If M_* is a graded R-module, we can construct $K(M_*, p)$ in a similar way such that

$$\pi_{s,t} K(M_*, p) = \begin{cases} M_t & s = p, \\ 0 & \text{else}. \end{cases}$$

Then

<u>Cor</u> $K(M_*, p) = \prod_q K(M_q, p, q)$.

(Moduli) problem: Find all complexes of R-modules with given homology.

M_* graded R-mod.

$$C_* \qquad \pi_k C_* \xrightarrow{\sim} M_k$$

$$\begin{array}{ccc} C_* \to C'_* & \qquad \pi_k C_* & \longrightarrow & \pi_k C'_* \\ & & \searrow^{\sim} \quad \swarrow^{\sim} & \\ & & M_k & \end{array}$$

Equivalently, write down all bi-cx. X such that

$$\pi_s \pi_t X = \begin{cases} M_t & s = 0, \\ 0 & s > 0. \end{cases}$$

(It is easy to see that these are indeed equivalent problems.)

1) Start with $K(M, 0)$. From the construction I, we see that

$$\pi_s \pi_t K(M,0): \qquad \begin{array}{c|cccc} t & \vdots & \vdots & & \\ & M_2 & M_3 & \cdots & \\ & M_1 & M_2 & \cdots & \\ & M_0 & M_1 & \cdots & \\ \hline & & & & s \end{array}$$

2) Let $\Omega M = M[1]$ be $M[1]_q := M_{q+1}$, and choose a map

$$K(M,0) \longrightarrow K(\Omega M, 2)$$

inducing an iso. on $\pi^h_* \pi^v_2$. The possible choices are parametrized by

$$\prod_k \mathrm{Ext}^1_R(M_k, M_{k+1}).$$

Take $P_0 = K(M,0)$, and, with the choice of map above, form the htpy fiber

$$\begin{array}{ccc} P_1 & & \\ \downarrow & & \\ P_0 & \longrightarrow & K(\Omega M, 2) \end{array}$$

We find

$$\left.\begin{array}{lcccc} \pi_*\pi_* P_0 & \underset{0}{M_*} & 0 & \Omega M_* & - - \\ & & & {\scriptstyle 2}\big\downarrow \wr & \\ \pi_*\pi_* K(\Omega M, 2) & 0 & 0 & \underset{2}{\Omega M_*} & \underset{3}{\Omega^2 M_*} \end{array}\right\} \Rightarrow$$

$$\begin{array}{lcccc} \pi_*\pi_* P_1 & \underset{0}{M_*} & 0 & 0 & \underset{3}{\Omega^2 M_*} \; - - \end{array}$$