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① Hopkins Miller Seminar

Algebraic Stacks and why Homotopy theorists are interested in them

$\{ \text{Schemes over } \mathbb{C} \} \xrightarrow[\text{Yoneda embedding}]{} \{ \text{contravar. functors Schemes} \to \text{Sets} \}$

just for our comfort

$X \longmapsto \mathrm{Hom}(-, X)$

Lemma (Yoneda): The Yoneda embedding is fully faithful.

Observation (Grothendieck): This is usefull.

Example: Functor of smooth cubic plane curves

$F: \; S \mapsto \{ \text{closed subschemes } Y \text{ of } S \times \mathbb{P}^2 \mid Y \text{ is flat over } S \text{ \& each fibre is a smooth cubic in } \mathbb{P}^2 \}$

"family of cubic curves parametrized by S"

functor by pullback.
This functor F is representable.

$\mathbb{C}^{10}$ ← homogeneous cubics in 3 variables

$\cup$

$\mathbb{C}^{10} \setminus \{0\} \longrightarrow \mathbb{P}^9$ represents all cubic curves on the plane

But not all of these are smooth

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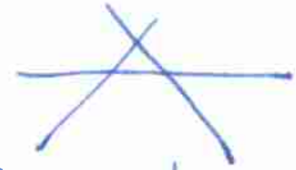
e.g. xyz



zero set

pts are singular

$U \subset \mathbb{P}^9$ subset corresponding to smooth cubics

(complement of hyperplane $\Rightarrow$ Zariski open)

U repr. F,

$F(S) = \mathrm{Hom}(S, U)$

in particular there is a universal cubic curve

$Y \subseteq U \times \mathbb{P}^2$.

[elliptic curve usually means genus one curve with basepoint, but we are going to be sloppy & ignore bpt & say ell. curve = gen. one curve]

[smooth cub. curves in plane are ell. curves]

Let's look at another moduli problem:

$$F(S) = \left\{ \begin{array}{c} E \\ \downarrow \pi \\ S \end{array} \;\middle|\; \begin{array}{l} \pi \text{ is proper, flat,} \\ \text{geom. fibres are connected} \\ \text{smooth genus 1 curves} \end{array} \right\} \Big/ \text{isoms over } S$$

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but we want

$$\begin{tikzcd} E \arrow[rr, phantom, "\simeq"] \arrow[dr] & & E' \arrow[dl] \\ & S & \end{tikzcd}$$

to be one point in moduli space

$$F(S) \xrightarrow[\text{j-invariant}]{} \mathbb{A}^1$$

Γ classifies ell. curve / field up to isom (alg. closed)

family version

$\rightsquigarrow$

$\Rightarrow$ $\mathbb{A}^1$ looks like a good candidate

but over general basis, this is no good.
no decent map in to other direction
"coarse moduli space"

Try to figure out what is really going wrong here:

$\{ - \} /$ isom is naive, since two ell. curves could be isom in different ways e.g. any curve has autom (-1)

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Think of top. space 
two pts in same path component. These two pts could really be interchanged, but doesn't make sense to collaps all the connected components to points.

Rather than mod out, look at the groupoid valued fctor

$$\mathcal{M}_1(S) = F(S) = \left\{ \begin{array}{c} E \\ \downarrow{\scriptstyle \pi} \\ S \end{array} \;\middle|\; \cdots \right\}$$

genus one

groupoid via

$$\begin{tikzcd} E' \arrow[rr, "\sim"] \arrow[dr] & & E \arrow[dl] \\ & S & \end{tikzcd}$$

$\{$schemes over $\mathbb{C}\}$ $\longrightarrow$ $\{$contravar. functors schemes $\to$ sets$\}$

$\cap$ $\qquad\qquad\qquad\qquad$ $\cap$

$\{$Artin Stacks$\}$ $\subseteq$ (fully faithfull) $\{$contravar. functors schemes $\to$ groupoids$\}$

there is no hope to represent such a thing by a scheme, but we would like some kind of geom. objects, Artin-stacks, that represent grpd's valued fcts

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any smooth cubic curve is an ell. curve $\Rightarrow$ the universal one gives an ~~map~~ element of $\mathcal{M}_1(U)$ or a map $U \xrightarrow{\varphi} \mathcal{M}_1$ (nat'l transfo of functors

what do fibres of φ look like?

$$\{\underset{\text{over } E}{\text{Fibre}}\} = \{\text{embeddings } E \hookrightarrow \mathbb{P}^2\}$$

$\mathcal{O}(1)$ line bundle on $\mathbb{P}^2$. $E \hookrightarrow \mathbb{P}^2$ gives $\mathcal{O}(1)|_E$ line bdl of deg 3 on E (since cubic embedding)

this corresponds to the fact that a line in $\mathbb{P}^2$ is going to meet a cubic curve in three points

$$\begin{array}{c} \text{Fibre} \\ \downarrow \\ \text{Space of line bdls of deg 3 on } E \end{array} \approx E$$

... torsor ...

...

what does a fibre of that map look like? Fix E & fix a line bdl $\mathcal{L}$ of deg 3 on E

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$\Gamma(E, \mathcal{L})$ = 3 dimensional space

choose a basis

to get $E \hookrightarrow \mathbb{P}^2$

$\Rightarrow$ fibre of that map is

$\text{fiber}|_{\mathcal{L}} = \{\text{bases for } \Gamma(E,\mathcal{L})\} / \mathbb{C}^\times$ ~~s.t. scaling by a~~

$\Rightarrow$ fibre smooth & nonempty

what that means is roughly

that $\mathcal{U} \longrightarrow \mathcal{M}_1$ is smooth

& surjective

Def: Artin stack are functors

F : schemes $\longrightarrow$ Gpds s.t.

$\exists$ affine scheme $\mathcal{U}$ & $\varphi : \mathcal{U} \to F$ s.th.

$\mathcal{U}$ is smooth & surjective.

(F = space) relatively representable

by maps that are smooth & surjective.

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How do we get a handle on this?

(7) What does smooth & surj. do for you?

X is a scheme means

$\exists$ $U_i \longrightarrow X$ open immersions

$\Vert$

$\operatorname{Spec} A_i$

U for Artin Stacks $\hat{=} \coprod U_i$

not open immersion but locally open immersion

$X \longleftarrow \coprod U_i$

$\{U_i \cong \operatorname{Spec} A_i\}$

$X = \coprod U_i \,/\,$ some kind of equiv. relu

so we just need to specify

Jacob $\mathcal{U} = \coprod \mathcal{U}_i \leftleftarrows R = \mathcal{U} \times_X \mathcal{U}$

8 $\quad X = \cancel{\mathcal{U}/R}$ coeq $(R \rightrightarrows \mathcal{U})$

for Artin Stack no longer equiv reln but "equiv. gpd"

$\mathcal{U} \times_{\mathcal{F}} \mathcal{U} \rightrightarrows \mathcal{U} \longrightarrow \mathcal{F}$ an Artin Stack

part of defn of being smooth & surjective is saying that $\mathcal{U} \times_{\mathcal{F}} \mathcal{U}$ is representable by a geom. obj. a little more general than scheme. But think scheme

algebraic space

$$\mathcal{U} \times_{\mathcal{F}} \mathcal{U} \underset{\overline{u}_2}{\overset{\overline{u}_1}{\rightrightarrows}} \mathcal{U}$$

$\Delta : \mathcal{U} \longrightarrow \mathcal{U} \times_{\mathcal{F}} \mathcal{U}$

swap : $\mathcal{U} \times_{\mathcal{F}} \mathcal{U} \longrightarrow \mathcal{U} \times_{\mathcal{F}} \mathcal{U}$

$m =$ mult : $\mathcal{U} \times_{\mathcal{F}} \mathcal{U} \times_{\mathcal{U}} \mathcal{U} \times_{\mathcal{F}} \mathcal{U} \longrightarrow \mathcal{U} \times_{\mathcal{F}} \mathcal{U}$

$(u_1, u_2) \quad (u_2, u_3) \longmapsto u_1, u_3$

identif. of their images in $\mathcal{F}$ — iden. of in $\mathcal{F}$ — composite of the identifications.

$(U, U \times_{\mathcal{F}} U)$ forms a groupoid

Jacob object in schemes.

9 conversely given these data, can recover Artin Stack (....)

Things simplify a little ~~is~~ if we ask that $U \times_{\mathcal{F}} U$ is affine

[actually you can always arrange that U be affine, replace non-aff X ~~is~~ by $\coprod U_i$ as above.
But in general not true for $U \times_{\mathcal{F}} U$]

However in the applications to htpy theory, often is. Assume therefore

$U = \operatorname{Spec} A$ $\qquad \pi_1^*, \pi_2^* : A \to B$

$U \times_{\mathcal{F}} U = \operatorname{Spec} B$ $\qquad \text{swap}^* : B \to B$

$\Delta^* : B \to A$

$m^* : B \to B \otimes_A B$

Hopf - algebroid.