Jacob

(10)

If B is nice as A algebra (for Artin stack : smooth) but this condition is too strong ~~(for applic. in htpy theory)~~ and for our purposes it is enough to assume <u>flat</u>.

If you want to do lots of geometry, you want it to be Artin stack. Apart from that maybe flat is enough

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Homotopy theory

Suppose that E is a (naively) commutative & associative ring spectrum.

$\pi_* E \rightrightarrows \pi_*(E \wedge E)$ ~~...~~ $\to E \wedge E$ is, too

assume flat

also assume that both of these are concentrated in even degrees (get rid of graded commutativity)

Jacob $\pi_* E \rightrightarrows \pi_* E \wedge E$

(11)

$$A \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} B \qquad \mathrm{swap}^* : B \to B$$

$$B \xrightarrow{\Delta^*} A \qquad m^* : B \to B \otimes_A B$$

uses flatness $\Rightarrow$

$$\pi_*(E \wedge E \wedge E)$$
$$\uparrow$$
$$\pi_*(E \wedge S^0 \wedge E)$$

$\rightsquigarrow$ Hopf algebroid.

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ring spectrum E $\rightsquigarrow$ Hopf algebroid $\rightsquigarrow$ groupoid object in schemes

$\wr$

Artin stack (almost)

Why is this useful?

ASS to compute some approximation of htpy gps of spheres

E_2-term turns out to be the cohomology of the structure sheaf of the stack. (absolutely clear from defn. but ignoring the grading)

E_r-term completely incomprehensible from algebraic geom. pt of view.

Dacos

Everybody's favourite example

$E = HU$

12

Quillen: $\mathrm{Hom}(\pi_* HU, A) = \{fgl\text{'s}/A\}$

$$\mathrm{Hom}(\pi_* HU, A) = \mathrm{Hom}(S, \mathrm{spec}(\pi_* HU)) = \{fgl\text{'s}/S\}$$

fgl over each affine piece & patch them together or formal scheme ~~of~~ over S with various properties

$$\mathrm{Hom}(\pi_*(HU \wedge HU), A)$$

$\|$

2 fgl's over A & a strict isom between them.

(in part. have same underlying formal gp)

$\rightsquigarrow$ get infinite dimensional Artin Stack

$HU_* HU$ over HU_* ~~has~~ is polyn. ring in infinitely many variables

"Artin stack"

$$\begin{tikzcd} \mathrm{Spec}\, HU_* HU \arrow[loop right] \arrow[d, shift left=2] \arrow[d, shift right=2] \\ \mathrm{Spec}\, HU_* \arrow[u] \end{tikzcd}$$

& mult

M: ~~M~~

M is the functor

(13) $S \longmapsto$ comm. one di- formal groups over S

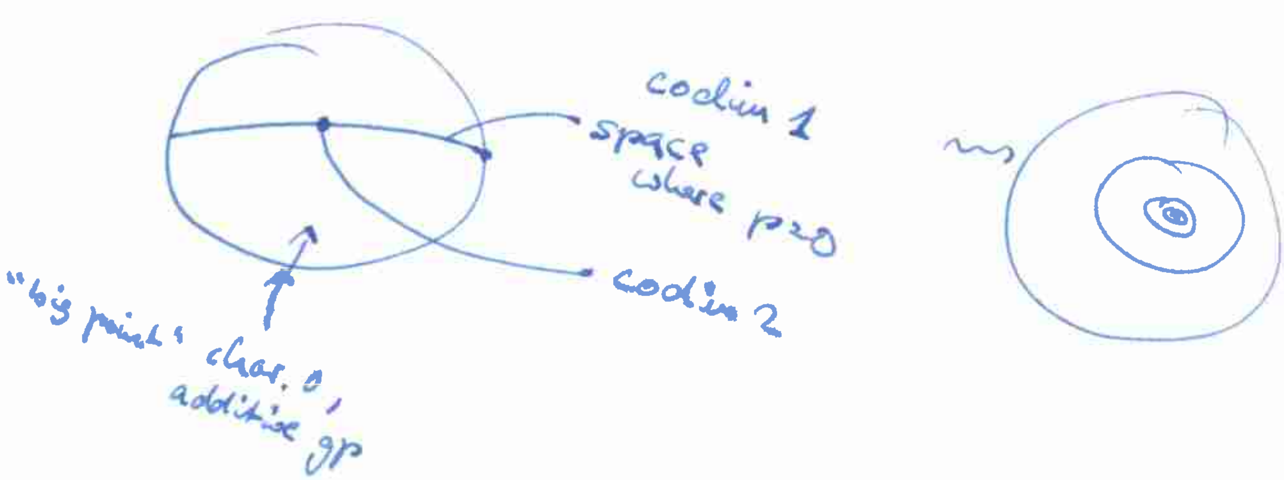
not quite right. but would be right if we would erase the word strict.

So really

$S \longmapsto$ comm. one dim f gps / S & (harmless) additional data (i.e. plus trivialization of ~~set~~ $\underline{\omega}$)

$\text{Dim } M = \infty - \infty = 0$

Assume we are over $\mathbb{Z}_{(p)}$



Jacobs $\{\circ\} = \bigcap$ all of these height ∞

$\circ - \mathbb{G}_a -$ in char. p

(14)

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each of these rings is a point

we say "codim 1" because it is

cut out by a single function.

"$\infty - \infty = 0$" b/c all pts.

$\boxed{\text{LEFT}}$ Landweber exact functor theorem

Suppose E_* is a graded module over MU_*

(work over $\mathbb{Z}_{(p)}$)

consider the functor

$$X \mapsto (MU_*(X)) \otimes_{MU_*} E_* =: E_*(X)$$

Spectrum ~~CW cx~~ finite

Question: is $E_*(X)$ a homology theory?

- $MU_*(-)$ homology theory. The only condition that we have to worry about is that certain sequences have to be exact:

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$$X' \to X \to X''$$

cofibre sequence (in finite spectra)

(15)

$$\dots \to MU_* X' \to MU_*(X) \to MU_*(X'') \to MU_{*-1} X' \to \dots$$

have exact sequence of graded vector spaces and tensor it with E_*. If e.g. E_* was flat over MU_*, that would work. But Landweber exact functor thm has much weaker condition.

Recall that a module E over a ring A

(*) is flat iff $\mathrm{Tor}_1^A(E, N) = 0$

$\forall$ finitely presented A-module.

Our moduli stack M receives a covering map

$$\mathrm{Spec}\, MU_* \longrightarrow M$$

Suppose X is any spectrum. $MU_* X$ is

Any quasi coherent sheaf on Spec MU_* arising as associated q. coh. sheaf of $MU_* X$ is the pullback of some

Jacob q.c. sheaf on M.

16

* This is algebraic geometer's language for saying that MU_*X is an (MU_*, MU_*MU) - comodule.

For Landweber's theorem, we need ~~not~~ only a weaker condition for (*)

$$\mathrm{Tor}_1^{MU_*}(E_*, N) = 0$$

only when N comes from moduli stack

Tor in category of A modules, since we are tensoring over MU_* (do) not take into account the MU_*MU -comod. operations

Suppose A is Noetherian, M finite A-module. Then $\exists$ filtration $0 = M_0 \subseteq \dots \subseteq M_n = M$ $M_k/M_{k-1} \cong A/p_i$. On Spec A any coherent sheaf has a filtration s.t. successive quotients are str. sheaves of reduced irred. subschemes

Jacob

(17)

Our stack : almost noetherian

dir lim of . . . .

"Coherent alg. stack"

not affine but that statement is still true. But that is very good, b/c we know what all the closed irred subschemes are

e.g. ~~loco~~ locus where $p = 0$

$\hookrightarrow$ locus where $v_1 = 0$

. . .

$M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_\infty$

$M_n = \{$ closed piece classifying formal groups of height $\geq n$ $\}$

Any coherent sheaf F admits a finite filtration

$0 = F_0 \subseteq \dots \subseteq F_k = F$ s.t. F_i / F_{i-1}

are vector bundles over M_i $i < \infty$

(trivial? I don't know. should be careful b/c of ex. of cohom.op's)

Jacob

18

each successive quotient as an MU_*-module, looks like a direct sum of $MU_*/(p, v_1, \dots, v_n)$

So in fact we only need

$\mathrm{Tor}_1^A(E, N)$ to vanish

for N coming from moduli stack & every such N can be filtered as above, so we only need to check this when $N = MU_*/(p, v_1, \dots, v_k)$.

So the Landweber exact functor thm says that $MU_*(X) \otimes_{MU_*} E$ is homology theory if $(p, v_1, \dots, v_k, \dots)$ is exact sequence on E.

e.g. $\mathrm{Tor}_1^{MU_*}(E, MU_*/p)$
$= p$-torsion elements of E

$\Rightarrow$ no p-torsion

$\mathrm{Tor}_1^{MU_*}(E, MU_*/p, v_1) = \dots$

v_1 needs to be injective on MU_*E