On complex numbers and *eigenstuff*

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**Definition 1.** Let \( z_1 = x_1 + iy_1 \), \( z_2 = x_2 + iy_2 \). We define

\[
z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2),
\]
\[
z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).
\]

Note that \( z_1z_2 \) is just the usual product of two binomials, where we grouped together all terms with \( i \) (and used that \( i^2 = -1 \)).

**Definition 2.** Let \( z = x + iy \). We define its real and imaginary parts as

\[
\Re z = x \quad \text{and} \quad \Im z = y.
\]

If \( z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n \), we can write

\[
v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + i \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},
\]

and define the real and imaginary parts of \( v \) as

\[
\Re v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \Im v = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.
\]

**Definition 3.** Let \( z = x + iy \). We define the *complex conjugate* of \( z \) as

\[
\bar{z} = x - iy.
\]

If \( A = (z_{ij}) \) is a matrix, we define its complex conjugate

\[
\bar{A} = (\bar{z}_{ij});
\]

i.e., the complex conjugate of a matrix (or a vector) is just the matrix (or vector) whose components are the complex conjugate of the components of the original matrix.

If \( x \) is a real number, \( x = x + 0i \), so \( \bar{x} = x \), and similarly for matrices with real entries. It can be shown, using Definition 1, that \( \bar{z_1 + z_2} = \bar{z_1} + \bar{z_2} \) and \( \bar{z_1z_2} = \bar{z_1}\bar{z_2} \). The same results hold when multiplying matrices.
**Definition 4.** The modulus, or absolute value, of \( z = x + iy \) is

\[
|z| = \sqrt{x^2 + y^2}.
\]

Note that \( |z|^2 = z \overline{z} \). This allows us to write, for \( z \neq 0 \),

\[
\frac{1}{z} = \frac{\overline{z}}{z \overline{z}} = \frac{\overline{z}}{|z|^2},
\]
a useful identity.

**Proposition 5.** Let \( A \) be a \( n \times n \) matrix with real coefficients. If \( \lambda \) is an eigenvalue of \( A \) with associated eigenvector \( v \), then \( \bar{\lambda} \) is also an eigenvalue, with associated eigenvector \( \bar{v} \).

**Proof.** Since \( v \) is an eigenvector of \( A \) associated to \( \lambda \),

\[
Av = \lambda v.
\]

Taking conjugate of this expression,

\[
\overline{A} \overline{v} = \overline{\lambda} \overline{v}
\]

But \( \overline{A} = A \), since the coefficients of \( A \) are real numbers:

\[
A \overline{v} = \overline{\lambda} \overline{v}.
\]

\( \square \)

**An example**

The eigenvalues and eigenvectors of \( A = \begin{bmatrix} -11 & 4 \\ -26 & 9 \end{bmatrix} \) are \( \lambda_1 = -1 + 2i \) and \( \lambda_2 = -1 - 2i \), with eigenvectors

\[
v_1 = \begin{pmatrix} 2 \\ 5 + i \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 2 \\ 5 - i \end{pmatrix},
\]

respectively. It is not difficult to see that \( \overline{\lambda}_1 = \lambda_2 \), and \( \overline{v}_1 = v_2 \).

Any nonzero constant multiple of \( v_1 \) is also an eigenvector associated to \( \lambda_1 \). In this context, however, “constant” means complex number. For instance, if we input \( A \) in Mathematica, it gives the eigenvector \( w_1 = ((5 - i)/13, 1)^T \) associated to \( \lambda_1 \). It turns out that \((5 + i)w_1 = v_1\).

The augmented coefficient matrix of the system \((A - \lambda_1 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\) is

\[
\begin{bmatrix}
-10 - 2i & 4 & 0 \\
-26 & 10 - 2i & 0
\end{bmatrix}
\]

Since \( |A - \lambda_1 I| = 0 \), it follows that \( A - \lambda_1 I \) is a noninvertible matrix, so the equations are redundant. We thus consider only the equation associated to the first row

\[
(-10 - 2i)x + 4y = 0.
\]

(1)
In class we solved for \( y \), and found \( y = \frac{10 + 2i}{4} \). Taking \( x = 2 \), we find \( y = 5 + i \), yielding the eigenvector \( v_1 \). We could have also solved for \( x \), yielding
\[
x = \frac{4}{10 + 2i} y = \frac{2}{5 + i} y.
\]
Taking \( y = 5 + i \), we recover \( v_1 \). If, instead, we consider the more “natural” choice \( y = 1 \), the resulting eigenvector is
\[
w_3 = \left( \begin{array}{c}
\frac{2}{5 + i} \\
1
\end{array} \right).
\]
If \( z = 5 + i \), it follows that
\[
\frac{2}{z} = \frac{2\bar{z}}{|z|^2} = \frac{2(5 - i)}{26} = \frac{5 - i}{13},
\]
so \( w_3 = w_1 \), though it’s not easy to see it.

If one is asked to find eigenvalues and eigenvectors of \( A \), in rigour giving the eigenvector \( w_3 \) associated to \( \lambda_1 \) is right. The main disadvantage of writing the eigenvector as \( w_3 \) is that
\[
\Re w_3 \text{ is NOT } \left( \begin{array}{l}
\frac{5}{5} \\
1
\end{array} \right), \quad \text{and} \quad \Im w_3 \text{ is NOT } \left( \begin{array}{l}
2 \\
0
\end{array} \right).
\]
To find \( \Re w_3 \) and \( \Im w_3 \) we must write \( w_3 \) as one real valued vector, plus \( i \) times another real valued vector. Using that
\[
w_3 = w_1 = \left( \begin{array}{l}
\frac{5 - i}{13} \\
1
\end{array} \right) = \left( \begin{array}{l}
\frac{5}{13} \\
1
\end{array} \right) + i \left( \begin{array}{l}
\frac{-1}{13} \\
0
\end{array} \right),
\]
we readily find
\[
\Re w_3 = \Re w_1 = \left( \begin{array}{l}
\frac{5}{13} \\
1
\end{array} \right), \quad \Im w_3 = \Im w_1 = \left( \begin{array}{l}
-\frac{1}{13} \\
0
\end{array} \right).
\]
Now, this was a lot of work just to find the real and imaginary parts of an eigenvector. It would have been much easier to do it with \( v_1 \):
\[
\Re v_1 = \left( \begin{array}{l}
\frac{2}{5} \\
0
\end{array} \right), \quad \Im v_1 = \left( \begin{array}{l}
0 \\
1
\end{array} \right).
\]
Why do we care about real and imaginary parts of eigenvectors? We will use them when solving linear systems of differential equations.

**TL;DR:** to make your life easier when finding complex eigenvectors, never divide by complex numbers; namely, in (1) you should solve for \( y \).

**Remark.** Note that if we had written \( w_1 = \left( \begin{array}{l}
\frac{5}{13} \\
1
\end{array} \right) - i \left( \begin{array}{l}
\frac{1}{13} \\
0
\end{array} \right) \), then the imaginary part would have been wrong. That’s why we’ll always write complex valued vectors as a real valued vector plus \( i \) times another real valued vector.