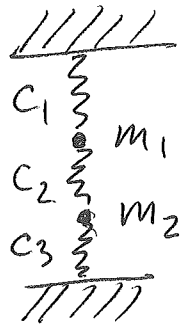
4. (15 points) Describe a mass-spring chain with both ends fixed that gives rise to the potential energy function

$$3u_1^2 - 4u_1u_2 + 3u_2^2 + u_1 - 3u_2$$

and find its equilibrium configuration.



$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad K = 2\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}, \quad \vec{f} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$p(\vec{u}) = \frac{1}{2}\vec{u}^T K \vec{u} - \vec{u}^T \vec{f}$$

$$K = A^T C A$$

$$\begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}^T \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} c_3 & 0 \\ 0 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \\ 0 & c_3 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 & 0 \\ 0 & c_2 - c_3 \end{pmatrix}$$

$$\begin{pmatrix} c_1 - c_2 & 0 \\ 0 & c_2 - c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -4 & 6 \end{pmatrix}$$

$$c_2 = 4, \quad c_1 = 2, \quad c_3 = 2$$

$$K\vec{u} = \vec{f} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \qquad \left(\begin{array}{cc|c} 6 & -4 & -1 \\ -4 & 6 & 3 \end{array}\right) \xrightarrow{\frac{2}{3}(1)+(2)} \left(\begin{array}{cc|c} 6 & -4 & -1 \\ 0 & \frac{10}{3} & \frac{7}{3} \end{array}\right)$$

$$\frac{10}{3}u_2 = \frac{7}{3} \qquad u_2 = \frac{7}{10}$$

$$6u_1 - \frac{14}{5} = -1$$

$$6u_1 = \frac{9}{5} \qquad u_1 = \frac{3}{10}$$

$$\vec{u} = \begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix}$$