Problem 1. Prove that
\[
\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.
\]

Problem 2. Prove that
\[
\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.
\]

Problem 3. Find \(\gcd(210, 48)\) in two different ways and find two integers \(s\) and \(t\) such that \(210s + 48t = \gcd(210, 48)\).

Problem 4. Find an integer \(x\) such that
\[
x \equiv 2 \pmod{5},
\]
\[
3x \equiv 1 \pmod{8}.
\]

Problem 5. Find \([20877 \mod 24]\) and \([20878 \mod 24]\) in \(\mathbb{Z}_{16}\). [Hint: Use Euler’s theorem.]

Problem 6. Is \(x^3 - 17x + 2\) irreducible in \(\mathbb{Z}[x], \mathbb{Q}[x], \mathbb{R}[x],\) and \(\mathbb{C}[x]\)? Explain why.

Solution: First of all, note that a polynomial of degree 3 factors in \(K[x]\), if and only if it factors as \((ax + b)(cx^2 + dx + e)\) in \(K[x]\). If \(K = \mathbb{Z}\), then since \(ae = 1\), \(a = \pm 1\), and we can assume without loss of generality that \(a = 1\) by moving the \(-1\) over to the second factor. If \(K = \mathbb{Q}, \mathbb{R},\) or \(\mathbb{C}\), we can move \(a\) over to the second factor and conclude that a polynomial of degree 3 is reducible in \(K[x]\), if and only if it has a factor \(x - b\). Thus, for any \(K\), a polynomial of degree 3 is reducible in \(K[x]\), if and only if it has a root in \(K\).

In \(\mathbb{Z}[x]\) and \(\mathbb{Q}[x]\): By one of the exercises, a rational root \(r/s\) in the reduced form will have \(r\vert 2\) and \(s\vert 1\). Thus we can assume \(s = 1\) and \(r = \pm 1\) or \(\pm 2\). Plug these into the polynomial, and we see that none of these is a root. Thus it has no rational and in particular no integral roots.

In \(\mathbb{R}[x]\): Call our polynomial \(p(x)\). Obviously, \(p(-10000) < 0\), whereas \(p(10000) > 0\). Since \(p(x)\) is continuous, by the Intermediate Value Theorem, it assumes all intermediate values, such as 0. Thus, it has a root, and therefore will be reducible over \(\mathbb{R}\).

In \(\mathbb{C}[x]\) every polynomial of degree at least 1 has a root and will thereby be reducible.

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Problem 7. List all motion symmetries of a regular pentagon.

Solution: Put the pentagon in the $xy$ plane with its centroid at the origin. Take the rotation $r$ of the pentagon about the $z$ axis by $360\degree/5 = 72\degree$, counterclockwise if you look from the above, and the $180\degree$ rotation $a$ about the axis passing through one of the vertices and the midpoint of the opposite side. These are symmetries of the pentagon. One can generate more symmetries by these:

$$\{e, r, r^2, r^3, r^4, a, ra, r^2a, r^3a, r^4a\}.$$ 

The rotations $e$, $r$, $r^2$, $r^3$, and $r^4$ are counterclockwise rotations by $0\degree$, $72\degree$, $2\cdot 72\degree$, $3\cdot 72\degree$, and $4\cdot 72\degree$, respectively, and rotate the space by different angles less than $360\degree$. Thus, they must be pairwise distinct. If $r^k a = r^m a$, then by multiplying by $a^{-1}$ on the right, we get $r^k = r^m$ with $0 \leq k, m \leq 4$, and we saw that all these are different for different $k$ and $m$. Finally, $r^k \neq r^m a$, because $r^m a$ flips the pentagon over and $r^k$ does not. Thus, we have got at least 10 distinct symmetries.

On the other hand, any symmetry of the pentagon must map a particular vertex, call it $A_1$, to another vertex, for which there are 5 different choices, and the counterclockwise adjacent vertex $A_2$ will be mapped to either of the nearby vertices to the vertex where $A_1$ went to. If we know where $A_1$ and $A_2$ go, we will know where the remaining vertices of the pentagon will go. There are $5 \cdot 2 = 10$ choices for that. Thus, there should be not more than 10 symmetries, and the above list shows all of them.