8.5 Applications to Differential Equations

1. (a) The Fourier cosine series for \( f(x) = \pi \) on \([0, 100]\) is
\[
50 - (400/\pi^2) \sum_{n=1}^{\infty} \cos(2n-1)(\pi x/100)/(2n-1)^2, \]
so the solution (8.35) of the heat equation is
\[
u(x, t) = 50 - (400/\pi^2) \sum_{n=1}^{\infty} \cos(2n-1)(\pi x/100)/(2n-1)^2.
\]
(b) When \( t = 60 \), the error in discarding the terms after \( m = 2 \) is
\[
\left| \frac{400}{\pi^2} \sum_{n=1}^{\infty} e^{-0.0066(2m-1)^2 \pi^2 \pi (2m-1)^2/\pi^2} \frac{1}{(2n-1)^2} \right| \leq \frac{400}{\pi^2} e^{-0.0066} \sum_{n=3}^{\infty} \frac{1}{(2n-1)^2} \approx 0.48.
\]
To within this error, \( u(x, 60) \approx 50 - (400/\pi^2) [e^{-0.0066 \pi^2} \cos(\pi x/100) + \frac{1}{2} e^{-0.0066} \pi^2 \cos(3\pi x/100)] \approx 50 - (37.97) \cos(\pi x/100) - (2.51) \cos(3\pi x/100) \), which is about 10 when \( x = 0, 12 \), when \( x = 10 \), and 40 when \( x = 40 \).

(c) For \( t \geq 3600 \), \( |u(x, t) - 50| \leq (400/\pi^2) e^{-0.0066} \sum_{n=1}^{\infty} 1/(2n-1)^2 = 50 e^{-0.396} \approx 1.0037 \).

Almost good enough, but not quite! A slightly less crude estimate works: \( |u(x, t) - 50| \leq (400/\pi^2) e^{-0.396} \sum_{n=1}^{\infty} 1/(2n-1)^2 \approx (400/\pi^2) e^{-0.396} \approx 0.81 \).

2. One follows the separation-of-variables procedure as on p. 382 to find solutions of the form \( e^{-k^2t} \left( C_0 \cos \sqrt{\alpha} \theta + C_2 \sin \sqrt{\alpha} \theta \right) \). The periodicity condition then forces \( \sqrt{\alpha} = n \pi \), so the resulting analog of (8.35) is \( u(\theta, t) = \sum_{n=0}^{\infty} e^{-k^2t} \pi^2 \frac{2}{\pi} \frac{\sin n \pi \theta}{\sin \pi \theta} \cos \pi n \theta \). To satisfy the initial condition one takes \( \sum_{n=0}^{\infty} (a_n \cos n \theta + b_n \sin n \theta) \) to be the Fourier series of \( f(\theta) \). (The result: looks a little neater in exponential form: \( u(\theta, t) = \sum_{n=0}^{\infty} c_n e^{-k^2t} n \pi^2 k^2 + i n \theta \) where \( f(\theta) = \sum_{n=0}^{\infty} c_n e^{in \theta} \).

3. If \( u(x, t) = \sum_{n=0}^{\infty} b_n(t) \sin n \pi x/l \) is to satisfy \( \partial_t u = k \partial^2_t u + G(t) \), where \( G(x, t) = \sum_{n=0}^{\infty} 2 \pi \sin n \pi x/l \), we must have \( b_n'(t) = -k \pi^2 b_n(t) + \beta(t) \), assuming that termwise differentiation of the series is justified. To solve this ordinary differential equation, multiply through by the integrating factor \( e^{k \pi^2 t} \) to obtain \( (d/dt) [b_n(t) e^{k \pi^2 t}] = c_n e^{k \pi^2 t} \beta(t) \), whence \( b_n(t) e^{k \pi^2 t} = b_n(0) + \int_0^t e^{-k \pi^2 s} \beta(t) \) ds. For this to work, the following conditions are (more) than sufficient: (1) \( f \) is of class \( C^1 \) on \([0, l]\) and \( f(0) = f(1) = 0 \). (2) \( G(x, t) \) is \( C^2 \) as a function of \( x \in [0, l] \) for each \( t \), \( G(0, t) = G(l, t) = 0 \), and \( G(x, t) \), \( \partial_x G(x, t) \), and \( \partial^2_x G(x, t) \) are jointly continuous as functions of \( x \in [0, l] \) and \( t \geq 0 \). The boundary conditions on \( f \) and \( G \) guarantee that their odd periodic extensions are still at least \( C^1 \), and that of \( \partial^2_x G \) is at least piecewise continuous. It follows that the Fourier sine coefficients of \( f \) (namely, \( b_n(0) \)) are absolutely summable, and those of \( G \) (namely, \( \beta(t) \)) are continuous in \( t \) and satisfy \( |\beta(t)| \leq C \pi^{-2} \) for \( t \) in any finite interval \([0, T] \). Then we have
\[
|b_n(t)| \leq e^{-k \pi^2 t} \left[ |b_n(0)| + C \pi^{-2} \int_0^t e^{-k \pi^2 s} \beta(s) \right] \leq e^{-k \pi^2 t} |b_n(0)| + \frac{C}{k \pi^2} \int_0^t \beta(s) \,
\]
This is enough to guarantee the absolute and uniform convergence of the series defining \( u(x, t) \) for \( x \in [0, l] \) and \( t \in [0, T] \), as well as the absolute and uniform convergence of the series defining \( \partial_t u(x, t) \) and \( \partial^2_x u(x, t) \) for \( x \in [0, l] \) and \( t \in [\epsilon, T] \) (\( \epsilon > 0 \)), so that all formal calculations are justified.

4. (a) The odd periodic extension of the initial displacement \( u(x, 0) \) is \( mg(\pi x/l) \) where \( g \) is as in Exercise 2, §8.2, with \( a = \pi b/l \), so its Fourier sine series can be read off from the answer to that exercise. The series for \( u(x, t) \) can then be read off from (8.37).

(b) When \( b = (0.1) \), we have \( 2l^2/\pi^2 (1 - b) = 200/24 \approx 200/9 \approx .844 \), and \( n^{-2} \sin((.4)n \pi) \approx .951, .147, -0.065, -0.059, 0 \) when \( n = 1, 2, 3, 4, 5 \), so the first five coefficients (up to the overall factor of \( m \)) are \( 200, 124, -0.055, -0.050, 0 \). When \( b = (0.1) \), we have \( 2l^2/\pi^2 (1 - b) = 200/9 \approx 2.252 \), and \( n^{-2} \sin((1.1)n \pi) \approx .309, .147, .090, .059, .040 \) when \( n = 1, 2, 3, 4, 5 \). So the first five coefficients are (in \( m \) times) \( .696, .331, .203, .133, .090 \). (Note: The \( L^2 \) norm of the initial displacement \( u(\cdot, 0) \) is \( m \sqrt{1/3} \), independent of \( b \), so the total energy of these waves is independent of \( b \) and a direct comparison of the coefficients is appropriate.)