Math 4606. Fall 2006.

Solutions to Exam 1

1. (20 points) Let \( X \) and \( Y \) be two non-empty sets and let \( f \) be a one-to-one function from \( X \) to \( Y \). Let \( A \) be a subset of \( X \). Show that \( f(X \setminus A) \) is a subset of \( Y \setminus f(A) \).

**Solution.** Let \( y \in f(X \setminus A) \), then \( y = f(x) \) for some \( x \in X \setminus A \). Suppose \( y \in f(A) \). Then there is \( x' \in A \): \( y = f(x') \). Since \( A \) and \( X \setminus A \) are disjoint, \( x \neq x' \). Therefore \( y = f(x) = f(x') \) with \( x \neq x' \) which contradicts \( f \) being one-to-one. Thus \( y \notin f(A) \), which means \( y \in Y \setminus f(A) \). Hence \( f(X \setminus A) \subset Y \setminus f(A) \).

2. (20 points) Let \( f \) be a function from \( \mathbb{R}^2 \) to \( \mathbb{R} \) given by

\[
f(x, y) = \begin{cases} 
2xy & \text{if } (x, y) \neq (0, 0), \\
0 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

Does the limit \( \lim_{(x,y) \to (0,0)} f(x, y) \) exist? Why? Find the limit if it does.

**Solution.** Let us look at \( \lim_{x \to 0} f(x, 0) \) and \( \lim_{x \to 0} f(x, x) \). The first limit exists and is equal to 0, because \( f(x, 0) = 0 \) for all \( x \). The second limit may be computed by computing the function \( f(x, x) \) for \( x \neq 0 \) as follows:

\[
f(x, x) = \frac{2x^2}{x^2 + 5x^4} = \frac{2}{1 + 5x^2}.
\]

Hence \( \lim_{x \to 0} f(x, x) = 2 \). Thus, we obtain two different limits as \( (x, y) \) approaches \((0, 0)\) along two different lines, which implies that \( \lim_{(x,y) \to (0,0)} f(x, y) \) does not exist.

3. (20 points) Let \( f, g \) and \( h \) be three real-valued functions on \( \mathbb{R}^n \) satisfying

\[ g(x) \leq f(x) \leq h(x) \text{ for all } x \in \mathbb{R}^n. \]

Let \( a \in \mathbb{R}^n \) and \( L \in \mathbb{R} \) and suppose that

\[ \lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L. \]

Prove that \( \lim_{x \to a} f(x) = L \).

**Solution.** Let \( \varepsilon > 0 \). Since \( \lim_{x \to a} g(x) = L \), there is \( \delta_1 > 0 \) such that

\[ -\varepsilon < g(x) - L < \varepsilon \text{ whenever } 0 < |x - a| < \delta_1. \]

Similarly, there is \( \delta_2 > 0 \) such that

\[ -\varepsilon < h(x) - L < \varepsilon \text{ whenever } 0 < |x - a| < \delta_2. \]
Let $\delta = \min\{\delta_1, \delta_2\}$. Let $x$ be in $\mathbb{R}^n$ such that $0 < |x - a| < \delta$. We have $0 < |x - a| < \delta_1$ and hence by (1):

$$f(x) - L \geq g(x) - L > -\varepsilon.$$  

We have $0 < |x - a| < \delta_2$ and hence by (2):

$$f(x) - L \leq h(x) - L < \varepsilon.$$  

Therefore $|f(x) - L| < \varepsilon$. Thus $\lim_{x \to a} f(x) = L$.

4. (20 points) Show that the set

$$S = \{(x, y) \in \mathbb{R}^2 : xy > 5 \text{ and } y + x^2 + 3x < 13\}$$

is an open set in $\mathbb{R}^2$.

Solution. We write $S = A \cap B$ where

$$A = \{(x, y) \in \mathbb{R}^2 : xy > 5\}, \quad B = \{(x, y) \in \mathbb{R}^2 : y + x^2 + 3x < 13\}.$$

Let $f_1(x, y) = xy$ and $f_2(x, y) = y + x^2 + 3x$. We know that $f_1$ and $f_2$ are continuous on $\mathbb{R}^2$. Since $A = f_1^{-1}((5, \infty))$ and $(5, \infty)$ is open, we obtain $A$ is open. Similarly, $B = f_2^{-1}((-\infty, 13))$ is open. Therefore $S$ is open, for being an intersection of two open sets.

5. (20 points) Find the limit

$$\lim_{k \to \infty} \frac{-3k^3 + 8k^2 - 7k + 11}{4k^3 - k^2 + 5}.$$  

Solution. Divide the denominator and numerator by $k^3$, we have

$$\frac{-3k^3 + 8k^2 - 7k + 11}{4k^3 - k^2 + 5} = \frac{a_k}{b_k},$$

where

$$a_k = -3 + \frac{8}{k} - \frac{7}{k^2} + \frac{11}{k^3}, \quad b_k = 4 - \frac{1}{k} + \frac{5}{k^3}.$$  

We have $\lim_{k \to \infty} a_k = -3$ and $\lim_{k \to \infty} b_k = 4 \neq 0$, hence

$$\lim_{k \to \infty} \frac{-3k^3 + 8k^2 - 7k + 11}{4k^3 - k^2 + 5} = \lim_{k \to \infty} \frac{a_k}{b_k} = -\frac{3}{4}.$$