Solutions for practice problems for the Final, part 3

Note: Practice problems for the Final Exam, part 1 and part 2 are the same as Practice problems for Midterm 1 and Midterm 2.

1. Calculate Fourier Series for the function $f(x)$, defined on $[-2, 2]$, where

$$f(x) = \begin{cases} -1, & -2 \le x \le 0, \\ 2, & 0 < x \le 2. \end{cases}$$

We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n x}{2} + b_n \sin \frac{\pi n x}{2} \right),$$

where

$$a_0 = \frac{1}{2} \left(\int_{-2}^{0} (-1)\, dx + \int_{0}^{2} 2\, dx \right) = 1,$$

$$a_n = \frac{1}{2} \left(\int_{-2}^{0} (-1) \cos \frac{\pi n x}{2}\, dx + \int_{0}^{2} 2 \cos \frac{\pi n x}{2}\, dx \right) =$$

$$\frac{1}{2} \left((-1) \left[\frac{2}{\pi n} \sin \frac{\pi n x}{2} \right]_{-2}^{0} + 2 \left[\frac{2}{\pi n} \sin \frac{\pi n x}{2} \right]_{0}^{2} \right) = 0, \quad n > 0,$$

and

$$b_n = \frac{1}{2} \left(\int_{-2}^{0} (-1) \sin \frac{\pi n x}{2}\, dx + \int_{0}^{2} 2 \sin \frac{\pi n x}{2}\, dx \right) =$$

$$\frac{1}{2} \left(-(-1) \left[\frac{2}{\pi n} \cos \frac{\pi n x}{2} \right]_{-2}^{0} - 2 \left[\frac{2}{\pi n} \cos \frac{\pi n x}{2} \right]_{0}^{2} \right) =$$

$$\frac{1}{\pi n}(1 - \cos \pi n) - 2 \frac{1}{\pi n}(\cos \pi n - 1) = \frac{3}{\pi n}(1 - (-1)^n).$$

Therefore, we have

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{3}{\pi n}(1 - (-1)^n) \sin \frac{\pi n x}{2}.$$

An easy way to see that all of a_n except a_0 are zero is to note that

$$f(x) = \frac{1}{2} + g(x),$$

where $g(x)$ is an odd function,

$$g(x) = \begin{cases} 3/2, & x > 0, \\ -3/2, & x < 0. \end{cases}$$

2. Calculate Fourier Series for the function $f(x)$, defined on $[-5, 5]$, where

$$f(x) = 3H(x-2).$$

By a similar method,

$$f(x) = \frac{9}{5} + \sum_{n=1}^{\infty} \left[\frac{-3}{\pi n} \sin\frac{2\pi n}{5} \cos\frac{\pi n x}{5} + \frac{3}{\pi n} \left(\cos\frac{2\pi n}{5} - (-1)^n \right) \sin\frac{\pi n x}{5} \right].$$

3. Calculate Fourier Series for the function, $f(x)$, defined as follows:

(a) $x \in [-4, 4]$, and

$$f(x) = 5.$$

Comparing $f(x)$ with the general Fourier Series expression with $L = 4$,

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{\pi n x}{4} + b_n \sin\frac{\pi n x}{4} \right),$$

we can see that $a_0 = 10$, $a_n = b_n = 0$ for $n > 0$ will give $f(x) = g(x)$.

(b) $x \in [-\pi, \pi]$, and

$$f(x) = 21 + 2\sin 5x + 8\cos 2x.$$

Again, for $L = \pi$, we have

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right),$$

and setting $a_0 = 42$, $a_2 = 8$, $b_5 = 2$ and the rest of the coefficients zero, we obtain $f(x) = g(x)$.

(c) $x \in [-\pi, \pi]$, and

$$f(x) = \sum_{n=1}^{8} c_n \sin nx, \quad \text{with } c_n = 1/n.$$

Similarly, we set $b_n = 1/n$ for $1 \le n \le 8$, and the rest of the coefficients zero.

(d) $x \in [-3, 3]$, and

$$f(x) = -4 + \sum_{n=1}^{6} c_n(\sin(\pi n x/3) + 7\cos(\pi n x/3)), \quad \text{with } c_n = (-1)^n.$$

We have

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n x}{3} + b_n \sin \frac{\pi n x}{3} \right),$$

so we set $a_0 = -8$, $a_n = 7(-1)^n$ for $1 \le n \le 6$ and $b_n = (-1)^n$ for $1 \le n \le 6$, and the rest of the coefficients zero.

4. (a) Let $f(x) = x + x^3$ for $x \in [0, \pi]$. What coefficients of the Fourier Series of f are zero? Which ones are non-zero? Why?

   $f(x)$ is an odd function. Indeed,

$$f(-x) = -x + (-x)^3 = -x - x^3 = -(x + x^3) = -f(x),$$

   therefore $a_n = 0$, and b_n can be nonzero.

   (b) Let $g(x) = \cos(x^5) + \sin(x^2)$. What coefficients of the Fourier Series of g are zero? Which ones are non-zero? Why?

   $g(x)$ is an even function. Indeed,

$$g(-x) = \cos((-x)^5) + \sin((-x)^2) = \cos(-x^5) + \sin(x^2) = \cos(x^5) + \sin(x^2) = g(x).$$

   Therefore, $b_n = 0$, and a_n can be nonzero.

5. Let $f(x) = 2x + x^4$ for $x \in [0, 5]$.

   (a) Write down the function $G(x)$, which is the odd continuation for $f(x)$. Specify what terms will be zero and non-zero in the Fourier expansion for $G(x)$.

   We have

$$G(x) = \begin{cases} 2x + x^4, & x > 0, \\ 2x - x^4, & x < 0. \end{cases}$$

Indeed, we can check that if $\alpha > 0$, then $G(-\alpha) = -2\alpha - (-\alpha)^4 = -2\alpha - \alpha^4 = -G(\alpha)$. In the Fourier expansion for G, $a_n = 0$, and b_n can be nonzero.

(b) Write down the function $V(x)$, which is the even continuation for $f(x)$. Specify what terms will be zero and non-zero in the Fourier expansion for $V(x)$.

We have

$$V(x) = \begin{cases} 2x + x^4, & x > 0, \\ -2x + x^4, & x < 0. \end{cases}$$

Indeed, we can check that if $\alpha > 0$, then $V(-\alpha) = -2(-\alpha) + (-\alpha)^4 = 2\alpha + \alpha^4 = V(\alpha)$. In the Fourier expansion for V, $b_n = 0$, and a_n can be nonzero.

6. Suppose $f(x)$ is defined for $x \in [0, 7]$, and $f(x) = 2e^{-4x}$. Another function, $F(x)$, is given by the following:

$$F(x) = \sum_{n=0}^{\infty} a_n \cos(\pi n x/7),$$

where

$$a_n = \frac{2}{7} \int_0^7 2e^{-4x} \cos\left(\frac{\pi n x}{7}\right) dx.$$

What is the value of $F(3)$? What is the value of $F(-2)$?

The function $F(x)$ is the cosine Fourier expansion of f. On the domain of f, that is, for $x \in [0, 7]$, we have $F(x) = f(x)$. Therefore, since $3 \in [0, 7]$, then $F(3) = f(3) = 2e^{-12}$.

For the negative values of x, the cosine series converges to the even extension of $f(x)$, which is $2e^{-4|x|}$. Therefore, $F(-2) = f(2) = 2e^{-8}$.

Note: a sine Fourier series would give the odd extension, and in this case we would have $-f(2) = -2e^{-8}$.

7. Let us supposed that both ends of a string of length 25 cm are attached to fixed points at hight 0. Initially, the string is at rest, and has the shape $4\sin(2\pi x/25)$, where x is the horizontal coordinate along the string, with zero at the left end. The speed of wave propagation along

the string is $3\,cm/sec$. Write down the complete initial and boundary value problem for the shape of the string.

We have the following initial and boundary value problem:

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}, \quad x \in [0,25], \tag{1}$$
$$y(0,t) = y(25,t) = 0, \tag{2}$$
$$y(x,0) = 4\sin(2\pi x/25), \tag{3}$$
$$\frac{\partial y(x,0)}{\partial t} = 0. \tag{4}$$

8. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = 50\frac{\partial^2 y}{\partial x^2}, \quad x \in [0,100], \tag{5}$$
$$y(0,t) = y(100,t) = 0, \tag{6}$$
$$y(x,0) = x^2(100-x), \tag{7}$$
$$\frac{\partial y(x,0)}{\partial t} = \begin{cases} x, & 0 \le x \le 25, \\ 1/3(100-x), & 25 \le x \le 100. \end{cases} \tag{8}$$

What is the speed of wave propagation along the string? What is the initial displacement of the string at point $x = 20$? What is the initial velocity of the string at point $x = 50$? At what point of the string is the initial velocity the largest?

The speed of wave propagation along the string is $\sqrt{50}$. The initial displacement of the string at point $x = 20$ is $20^2(100-20) = 32000$. The initial velocity of the string at point $x = 50$ is $1/3(100-50) = 50/3$. The maximum of the initial velocity is at point $x = 25$ (plot the graph of the initial velocity, equation (8), to see this).

9. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad x \in [0,2], \tag{9}$$
$$y(0,t) = y(\pi,t) = 0, \tag{10}$$
$$y(x,0) = 0, \tag{11}$$
$$\frac{\partial y(x,0)}{\partial t} = g(x). \tag{12}$$

Suppose that

$$\int_0^2 g(x) \sin\left(\frac{\pi n x}{2}\right)\, dx = \frac{1}{n^3}.$$

Find $y(x,t)$.

For the problem with the zero initial displacement, the solution is given in terms of the initial velocity (here $c = 1$),

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2} \sin\frac{n\pi t}{2},$$

with

$$c_n = \frac{2}{n\pi} \int_0^2 g(x) \sin\left(\frac{\pi n x}{2}\right)\, dx = \frac{2}{n^4 \pi}.$$

10. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad x \in [0,\pi], \tag{13}$$

$$y(0,t) = y(\pi,t) = 0, \tag{14}$$

$$y(x,0) = 22\sin 2x + 8\sin 6x, \tag{15}$$

$$\frac{\partial y(x,0)}{\partial t} = 0. \tag{16}$$

Find $y(x,t)$, in a closed form (containing no integrals). You will not need to evaluate any integrals.

We look for the solution is the form,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt.$$

To satisfy initial condition (15), we set $t = 0$ and obtain,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin nx.$$

To make this equal to $f(x) = 22\sin 2x + 8\sin 6x$, we set $c_2 = 22$, $c_6 = 8$, and the rest of them zero. We obtain,

$$y(x,t) = 22\sin 2x \cos 2t + 8\sin 6x \cos 6t.$$