MATH 5335: GÉOMÉTRIE UNE
A SHORT SOLUTION TO PROBLEM 3.8.26

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There is a shorter solution of Problem 3.8.26, but harder to arrive at. It is motivated by change of coordinates, of which we know only a general idea from short, descriptive sections, such as Section 3.7, of the text. Otherwise, this solution will look like it is based more on a trick, rather than on common sense logic, whereas the solution I presented in class was based on common sense, even though it was quite long. Anyway, recall that we have decided to take our isometry $\mathcal{U}(X)$ to be the counterclockwise rotation about point $(3, 2)$ by $\pi/2$. For each vector X, what does $\mathcal{U}$ do with the point $X + (3, 2)$? That is right, it rotates it by 90 degrees about $(3, 2)$, like any other point in the plane. The matrix of such a rotation about the origin would be

$$N = \begin{bmatrix} \cos\pi/2 & -\sin\pi/2 \\ \sin\pi/2 & \cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The result will be a vector pointing out from $(3, 2)$, so the resulting point will be $N[X] + [(3, 2)]$. Thus, we get the following formula:

$$\mathcal{U}(X + (3, 2)) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [X] + \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We know that, as any isometry, $\mathcal{U}(Y) = M[Y] + [P]$ for some matrix M and vector P. Plugging in $X + (3, 2)$ for Y, we get

$$M \left([X] + \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) + [P] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [X] + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and

$$M[X] + M \begin{bmatrix} 3 \\ 2 \end{bmatrix} + [P] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [X] + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

for any X. Thus, we will have

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$[P] = -M[(3, 2)] + [(3, 2)] = -[(-2, 3)] + [(3, 2)] = [(5. - 1)].$$

Finally,

$$U(X) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [X] + \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

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