Let $\alpha : I \rightarrow \mathbb{R}^2$ be a regular parameterized plane curve and $N(t)$ and $\kappa(t)$ be the normal vector and the curvature of $\alpha$, respectively. Assume $\kappa(t) \neq 0$ for all $t \in I$. Recall that in this situation the curve $E(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$ is called the evolute of $\alpha$. Show that the tangent line of the evolute is the normal line to $\alpha$ at $t$.

Show that the knowledge of the vector function $B(s)$ (the binormal vector) of a curve $\alpha$, with nonzero torsion everywhere, determines the curvature $\kappa(s)$ and the absolute value of the torsion $\tau(s)$ of $\alpha$.

One way to define a coordinate patch for the sphere $S^2$, given as $x^2 + y^2 + (z - 1)^2 = 1$, is to consider the so-called stereographic projection $\pi : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$ which carries a point $p = (x, y, z)$ on the sphere minus the north pole $N = (0, 0, 2)$ onto the intersection of the $xy$-plane with the straight line which connects $N$ to $p$. Let $(u, v) = \pi(x, y, z)$. (1) Show that $\pi^{-1} : \mathbb{R}^2 \rightarrow S^2$ is given by

$$x = \frac{4u}{u^2 + v^2 + 4}, \quad y = \frac{4v}{u^2 + v^2 + 4}, \quad z = \frac{2(u^2 + v^2)}{u^2 + v^2 + 4}.$$ 

(2) Show it is possible to cover the sphere with two coordinate patches.

Let $\lambda_1, \ldots, \lambda_m$ be the normal curvature at $p \in M$ along unit directions making angles $0, 2\pi/m, \ldots, (m - 1)2\pi/m$ with a principal vector, $m > 2$. Prove that $\lambda_1 + \cdots + \lambda_m = mH$, where $H$ is the mean curvature at $p$. [Hint: Use the fact that for $\theta = 2\pi/m$ $1 + \cos^2 \theta + \cdots + \cos^2(m - 1)\theta = \frac{m^2}{2}$.]

Determine the umbilic points of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. [Remark: I want to remove this problem from the sample exam. This problem is a good problem for a homework, but bad for an exam, because the computations involved are too long. A straightforward way to do it]
would be to compute all those $E,F,G,l,m,n$, as it is done in Example 2.3 on p. 95 of the text, then compute $K$ and $H$ and solve the equation $H^2 = K$, which is equivalent for a point to be umbilic, see Exercise 1.5 on p. 90.

A second, more efficient, but still too long, way would be to notice that the vector $N_1 = (x/a^2, y/b^2, z/c^2)$ is normal, therefore it is equal to $fN$, for a unit normal $N$ and $f = |N_1|$. Then observe that for any curve $\alpha(t) = (x(t), y(t), z(t))$ on the ellipsoid, a point is umbilic, if and only if it satisfies the equation

$$\left( \frac{dN_1}{dt} \times \frac{d\alpha}{dt} \right) \cdot N_1 = 0.$$  

Multiply this equation by $z/c^2$ and express $z$ and $zz'/c^2$ through $x, y, x', y'$. Then use the fact that the obtained equation should be satisfied for arbitrary $x'$ and $y'$, which gives a system of equations for $x$ and $y$. There will be 12 solutions.]