SOLUTION OF PROBLEM 10 FROM THE SECOND SAMPLE SET

Since the curve is a geodesic, its unit normal $N$ is equal to the unit normal $U$ to the surface. (In general, this is true up to a sign, but we can change the orientation on the surface, to make $N = U$.) Using one of the Frenet formulas, we get $U'' = N' = -\kappa T + \tau B$. Thus $U' \cdot U'' = \kappa^2 + \tau^2$. Let us compute $U' \cdot U''$, using the first and the second fundamental forms. If we show $U' \cdot U'' = -K$, we are done.

Use the chain rule: $U' = U_u u' + U_v v'$. We computed the partials $U_u$ and $U_v$ (which were interesting to us, because they are equal to $-S_p(x_u)$ and $-S_p(x_v)$, respectively), when we obtained the Weingarten equations and the formulas for the mean and the Gaussian curvatures through the first and the second fundamental forms. There we introduced a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

so that

$$U_u = a_{11} x_u + a_{21} x_v,$$
$$U_v = a_{12} x_u + a_{22} x_v.$$  

In other words, this matrix was defined by these equations. In the matrix form, this equation may be rewritten as follows.

$$\begin{pmatrix} U_u \\ U_v \end{pmatrix} = A^T \begin{pmatrix} x_u \\ x_v \end{pmatrix}.$$  

Our chain rule above in the matrix form looks like

$$U' = (U_u \quad U_v) \begin{pmatrix} u' \\ v' \end{pmatrix}. $$

Then

$$U' \cdot U' = (U')^T U' = (u' \quad v') \begin{pmatrix} U_u & U_v \\ U_u & U_v \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$= (u' \quad v') A^T \begin{pmatrix} x_u \\ x_v \end{pmatrix} A \begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$= (u' \quad v') A^T I A \begin{pmatrix} u' \\ v' \end{pmatrix},$$

where $I$ is the first fundamental form

$$I = \begin{pmatrix} x_u \\ x_v \end{pmatrix} \begin{pmatrix} x_u & x_v \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$  

We computed this matrix $A$ by obtaining the equation

$$-II = A^T I,$$

where

$$II = \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$  

is the second fundamental form. Thus we can replace $A^T I$ by $-\Pi$ in the equation for $U' \cdot U'$ above:

$$U' \cdot U' = -(u' \quad v') \Pi A \begin{pmatrix} u' \\ v' \end{pmatrix}.$$ 

Since the fundamental forms are symmetric, $\Pi = \Pi^T = -I^T A = -I A$, and we continue as follows.

$$U' \cdot U' = (u' \quad v') I A^2 \begin{pmatrix} u' \\ v' \end{pmatrix}.$$ 

Now recall that $A^2 - 2HA + K I d = 0$ (the characteristic equation of the matrix $A$). The surface is minimal, so $H = 0$ and we have $A^2 = -K I d$. Finally,

$$U' \cdot U' = -K (u' \quad v') I \begin{pmatrix} u' \\ v' \end{pmatrix}.$$ 

But $x' = x_u u' + x_v v'$ or in the matrix form

$$x' = (x_u \quad x_v) \begin{pmatrix} u' \\ v' \end{pmatrix}$$

and

$$x' \cdot x' = (x')^T x' = (u' \quad v') I \begin{pmatrix} u' \\ v' \end{pmatrix}.$$ 

$x' \cdot x' = 1$, because $x(s)$ is the equation of the geodesic, which we parameterized by arclength, as usually. Thus,

$$U' \cdot U' = -K.$$