SAMPLE MIDTERM EXAM II - MATH 5378, SPRING 2013

THIS WILL BE A CLOSED-BOOK, CLOSED-NOTES EXAM. YOU CAN USE IN YOUR SOLUTIONS ANY RESULT THAT WAS COVERED IN CLASS OR BY THE TEXT. YOU CAN ALSO USE ANY OF THE RESULTS FROM THE HOMEWORK.

Complicated formulas, such as the geodesic equations

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0,$$
$$v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0,$$

will be provided on the exam, if needed.

(1) Show that the surface (the conoid)

$$x(u,v) = (u\cos v, u\sin v, u+v)$$

is locally isometric to the surface (the hyperboloid of revolution)

$$y(s,t) = (s\cos t, s\sin t, \sqrt{s^2-1}),$$

by a mapping given by

$$t = v + \arctan u, \quad s^2 = u^2 + 1.$$

(2) Prove that a regular curve on a surface is a geodesic, if and only if its curvature is equal to the absolute value of its normal curvature.

(3) Two surfaces are tangent to each other along a common curve α. Prove that if α is a geodesic on one surface, then it will also be one on the other.

(4) **Please, disregard this problem, because it is too complicated for an exam, even though it is solved on p. 189: the metric coefficients and therefore the Christoffel symbols and the geodesic equations for the conic surface are just the same as the ones for the plane with polar coordinates. However, the problem is solved on p. 189 in a somewhat illegal way, because one needs to set the determinant from p. 188 to zero, rather than use the geodesic equations, which have $u = 0$ as the only solution, if you look for it in the form** $u = u(v)$. Find the geodesics of a conic surface $x(u,v) = u\alpha(v)$, where $\alpha(s)$ is a curve in space with $||\alpha|| = 1$ and $||\alpha'|| = 1$.

(5) Find the geodesic curvature of a helix $u = b$ on the helicoid $x = (u\cos v, u\sin v, av)$. Use the formula on p. 188 and an equation determining Γ^1_{22} on p. 174.

(6) A spherical bigon is a region on the unit sphere formed by two great semicircles with common endpoints. Find the area of a spherical bigon having an interior angle α by the vertex.

(7) What will be the Euler characteristic of a surface triangulated by 16 triangles so that each vertex is adjacent to 8 of them? If the surface is a sphere with g handles, what could g be equal to?

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(8) Express the sum of interior angles of a convex (*i.e.*, each interior angle being less than π) geodesic quadrangle on the sphere of radius R through the area of the quadrangle.
(9) Compute the Christoffel symbols for the metric $ds^2 = f(u,v)(du^2 + dv^2)$. You may use the six equations on page 174 just above *Theorema Egregium.*
(10) Prove that if a geodesic on a surface of revolution meets all parallels at a constant angle, the geodesic must be a parallel, a meridian, or the surface must be the upright cylinder.
(11) Show that the length of a line $x^2 + y^2 = 1$ on the Poincaré upper-half plane $y > 0$ is infinite.