Complicated formulas, such as the geodesic equations
\[ u'' + \Gamma^{1}_{11}(u')^2 + 2\Gamma^{1}_{12}u'v' + \Gamma^{1}_{22}(v')^2 = 0, \]
\[ v'' + \Gamma^{2}_{11}(u')^2 + 2\Gamma^{2}_{12}u'v' + \Gamma^{2}_{22}(v')^2 = 0, \]
and the Gauss and Mainardi-Codazzi equations (see p. 178 of the text), will be provided on the actual exam.

(1) Assume \( \alpha(s) \) is a unit speed curve of nonzero constant curvature whose normal lines are all perpendicular to a given constant vector. Show that \( \alpha \) is part of either a circle or a standard circular helix, which is a curve \( \beta(t) \) such that, after a Euclidean motion of the space, may be represented by an equation
\[ \beta(t) = (a \cos t, a \sin t, bt), \] for \( a \neq 0, b \neq 0. \)

(2) Show that a space curve with \( \kappa(s) \neq 0, \tau(s) \neq 0 \) and \( \kappa'(s) \neq 0 \) for all \( s \) lies on a sphere if and only if
\[ \frac{(1}{\kappa})^2 + \left( \frac{1}{\kappa} \right)^2 = \text{const}. \]

(3) Compute the first fundamental form of the following surfaces:
(a) \( x(u, v) = \left( \frac{u}{2} (u + 1), \frac{v}{2} (u - 1), v \right) \) (the hyperbolic cylinder);
(b) \( x(s, t) = \alpha(s) + sB(s) \) (the binormal surface of a curve unit-speed curve \( \alpha(s) \) with a binormal \( B(s) \)).

(4) Show that a compact, regular, oriented surface \( M \) in \( \mathbb{R}^3 \) has a point with \( K > 0 \). Furthermore, show that if \( M \) is not homeomorphic to a sphere, then it also has a point with \( K < 0 \) and a point with \( K = 0 \).

(5) Show there is no surface in \( \mathbb{R}^3 \) with \( E = G = c = 1, F = f = 0 \) and \( g = -1 \).

(6) Let \( \beta : I \rightarrow \mathbb{R}^3 \) be a curve of nonzero curvature and let \( x(u, v) = \beta(u) + v\beta'(u) \) be its tangent surface. Show that it is locally isometric to a cone (without the vertex).

(7) Compute the second fundamental form, the Gaussian and the mean curvatures of the following surface:
\[ xyz = a^3, \ a \neq 0. \]

(8) Find the angle through which the tangent vector to a closed curve \( \alpha \) on the unit sphere \( x^2 + y^2 + z^2 = 1 \) turns, if
(a) \( \alpha \) is a parallel;
(b) \( \alpha \) consists of two meridians different by an angle \( \phi \) and the part of the equator between them.

(9) Prove that on a surface of revolution, a meridian is always a geodesic directly, without using Clairaut’s relation. Assume the equation of a general
surface of revolution given, as well as the formulas for the Christoffel symbols of a surface of revolution and the geodesic equations.

(10) Let $M$ be a surface in $\mathbb{R}^3$ such that all its geodesics are plane curves. Show that $M$ is part of either a plane, or a sphere.

(11) Find geodesics on a general cylindrical surface

$$x(u, v) = (f(u), g(u), v),$$

where $u$ is the arclength parameter of the base curve $(f(u), g(u))$.

(12) Show that the equations $x_1^2 + x_2^2 = 1$ and $x_3^2 + x_4^2 = 1$ in $\mathbb{R}^4$ define a smooth manifold of dimension 2.