SELECTED SAMPLE FINAL EXAM SOLUTIONS - MATH 5378, SPRING 2013

Problem (1). **This problem is perhaps too hard for an actual exam, but very instructional, and simpler problems using these ideas will be on the actual exam.**

Since you are asked to show something up to Euclidean motion, it could hint on using the Fundamental Theorem for space curves. And that is what we are going to do.

Let $\mathbf{C}$ be the vector to which the normal $N(s)$ is perpendicular:

$$\langle N(s), \mathbf{C}\rangle = 0.$$

We have $\langle T(s), \mathbf{C}\rangle' = \langle \kappa N(s), \mathbf{C}\rangle = 0$. Thus, $\langle T(s), \mathbf{C}\rangle = b$, some constant. Similarly, using the Frenet-Serret $B' = -\tau N$, we get $\langle B(s), \mathbf{C}\rangle = a$, another constant. Differentiating the equation $\langle N(s), \mathbf{C}\rangle = 0$ and using the Frenet-Serret formula, we get

$$-\kappa b + \tau(s) a = 0.$$

We would love to conclude that $\tau(s)$ is constant, but not just yet: we need to know that $a \neq 0$ for that.

Suppose $a = 0$. Then $b = 0$ and $\langle T(s), \mathbf{C}\rangle = \langle N(s), \mathbf{C}\rangle = \langle B(s), \mathbf{C}\rangle = 0$, which implies that $\mathbf{C} = 0$, because T, N, and B, make up a basis (i.e., are linearly independent), given that κ is nonzero. In this case, any vector is perpendicular to $\mathbf{C}$, and the problem does not make sense.

Thus, $a \neq 0$, and $\tau(s) = \kappa b/a$ is a constant. If $b = 0$, then $\tau = 0$, whence $B' = -\tau N = 0$ or B is constant, and $\alpha(s)$ must be planar by Proposition 6.2. Now, a planar circle of radius R has constant curvature $\kappa = 1/R$. Thus, by the Fundamental Theorem for plane curves, there is a Euclidean motion of that plane that takes α to a circle of radius $1/\kappa$.

The remaining case is when $a \neq 0$ and $b \neq 0$. We want to show that up to Euclidean motion of space, α is a helix. The helix $\beta(t) = (c\cos t, c\sin t, dt)$, for $c > 0, d \neq 0$, has *constant* curvature $\frac{c}{c^2+d^2}$ and *constant* torsion $\frac{d}{c^2+d^2}$ (a direct computation, like at the bottom of p. 101). Thus, again using the Fundamental Theorem, but this time for space curves, we just need to see that given two constants $\kappa > 0$ and $\tau \neq 0$, we can find $c > 0$ and $d \neq 0$ to get $\kappa = \frac{c}{c^2+d^2}$ and $\tau = \frac{d}{c^2+d^2}$. You can solve this system of equations in many ways. One might be as follows: note that if (c, d) is a solution then $\kappa^2 + \tau^2 = (c^2 + d^2)^{-1}$. Therefore, $c = \kappa(c^2 + d^2) = \frac{\kappa}{\kappa^2+\tau^2}$ and $d = \frac{\tau}{\kappa^2+\tau^2}$. Thus, the helix with these c and d will have the required curvature and torsion.

Problem (2). **This problem might also be too hard for an actual exam, but very instructional, and simpler problems using these ideas will be on the actual exam.**

Date: May 12, 2013.

If an arc length parametrized curve $\alpha(s)$ lies on a sphere of radius R centered at C, we will have

$$\langle \alpha(s) - C, \alpha(s) - C\rangle = R^2.$$

If differentiate this equation, we will get

$$\langle \alpha(s) - C, T\rangle = 0.$$

Differentiate it once again:

$$1 + \kappa\langle \alpha(s) - C, N\rangle = 0. \tag{1}$$

Keep differentiating:

$$\kappa'\langle \alpha(s) - C, N\rangle + \kappa\langle \alpha(s) - C, N'\rangle = 0$$

and use the Frenet-Serret formula for N':

$$N' = -\kappa T + \tau B,$$

to get

$$\kappa'\langle \alpha(s) - C, N\rangle + \kappa\tau\langle \alpha(s) - C, B\rangle = 0$$

and therefore, using Equation (1),

$$-\frac{\kappa'}{\kappa} + \kappa\tau\langle \alpha(s) - C, B\rangle = 0$$

Now, we can compute the coefficients of $\alpha(s)-C$ in the Frenet-Serret basis $[T, N, B]$:

$$\begin{aligned} \alpha(s) - C &= \langle \alpha(s) - C, T\rangle T + \langle \alpha(s) - C, N\rangle N + \langle \alpha(s) - C, B\rangle B \\ &= -\frac{1}{\kappa}N + \frac{\kappa'}{\kappa^2\tau}B \\ &= -\frac{1}{\kappa}N - \left(\frac{1}{\kappa}\right)'\frac{1}{\tau}B. \end{aligned}$$

Since $\langle \alpha(s) - C, \alpha(s) - C\rangle = R^2$, we get

$$\frac{1}{\kappa^2} + \left(\left(\frac{1}{\kappa}\right)'\frac{1}{\tau}\right)^2 = R^2 = \text{const}\,. \tag{2}$$

Thus, we have shown that if the curve is on the sphere, then its curvature and torsion satisfy the required equation.

Conversely, suppose the curvature and torsion of a unit-speed curve satisfy this equation (2). We want to prove the curve is on the sphere. The solution above allows us to *guess* where the center and what the radius of that sphere should be:

$$C(s) := \alpha(s) + \frac{1}{\kappa}N + \left(\frac{1}{\kappa}\right)'\frac{1}{\tau}B, \tag{3}$$

$$R^2 := \frac{1}{\kappa^2} + \left(\left(\frac{1}{\kappa}\right)'\frac{1}{\tau}\right)^2. \tag{4}$$

Given the equation (2), we know that R defined just above must be a constant. And by construction, $R^2 = \langle \alpha(s) - C(s), \alpha(s) - C(s)\rangle$. The only thing we need to check is that $C(s)$ stays put in $\mathbb{R}^3$. The best way to do that is to see that the derivative of $C(s)$ vanishes. Indeed,

$$C'(s) = T + \left(\frac{1}{\kappa}\right)' N + \frac{1}{\kappa} N' + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)' B + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau} B'.$$

Now use the Frenet-Serret formulas to express N' and B' through T, N, and B and then differentiate the equation (4) to get

$$\frac{\tau}{\kappa} + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)' = 0$$

to see that all the terms in $C'(s)$ cancel and thereby $C'(s) = 0$.

Problem (3)**.** (a) $ds^2 = (a^2(1-u^{-2})^2 + b^2(1+u^{-2})^2)du^2 + dv^2$;
(b) $ds^2 = (1 + t^2\tau^2)ds^2 + dt^2$.

Problem (4)**.** Given that the surface M is compact, the distance from the origin to the point on M should achieve a maximum at some point $p \in M$. By construction, the surface will be inside the sphere of radius $R = ||p||$ centered at the origin. Therefore, every normal cross-section of M at p will lie in a circle of radius $r \leq R$ in that plane. We had a homework problem, in which we showed that the curvature k_n of such a curve must satisfy $|k_n| \geq 1/r \geq 1/R > 0$. In particular, the principal curvatures will satisfy these inequalities. Thus, all we know is that they are nonzero. However, the tangent plane to M at this point must be orthogonal to vector p, because otherwise some points on M near p will be outside of the sphere. If we choose the unit normal to M at p to point in the direction of p, then the normal vectors to normal cross-sections will point inside the sphere, as this is where the acceleration vectors of these cross-sections will point. In this case, all the k_n's will be negative, as the acceleration vectors will be opposite to the normal to the surface, and we get $k_n \leq -1/R$. Therefore, the product K of the principal curvatures will be $\geq 1/R^2 > 0$ at point p.

Since M is a compact and oriented surface, it must be homeomorphic to the sphere with g handles. Since M is not homeomorphic to S^2, we have $g > 0$ or, equivalently, $g \geq 1$. Thus, the Euler-Poincaré characteristic $\chi(M)$ will be $2-2g \leq 0$. By the Gauss-Bonnet theorem, we must have

$$\int\int_M K dA = 2\pi\chi(M) \leq 0.$$

But a neighborhood of the point p contributes to the double integral positively. Therefore, there must be a point $q \in M$ with $K < 0$ to compensate for that positive contribution. By continuity of K and path-connectedness of M, there will be a point on a path between p and q at which $K = 0$.

Problem (5)**.** Since $F = 0$, it is real easy to find the Christoffel symbols from the equations expressing them through the metric coefficients and their derivatives and see that the Christoffel symbols are all 0. The Gaussian curvature will have to be equal to -1, because the differential of the Gauss map will be $dN_p = -II_p I_p^{-1}$ (via the matrices of the second and first fundamental forms). This implies that dN_p is diagonal with 1 and -1 on the diagonal, whence $K = -1$. Then the first Gauss equation on p. 178 will turn into $0 = 1 \cdot (-1)$, which is a contradiction. Thus, such a surface will not exist.

Problem (6)**.** Compute the Gaussian curvatures for both surfaces, using the formula via the first and second fundamental forms, as in the previous problem (or using the determinant formula on p. 176). To simplify these computations, it will be a good idea to reparametrize the surface, using the arc length parameter for β, i.e., without loss of generality, we can assume that β' has norm one. And use the simplest cone $z^2 = x^2 + y^2$. The computations should show that $K = 0$ in both cases. Since the Gaussian curvatures are constant, Minding's Theorem 12.5 implies that the surfaces are locally isometric.

Problem (7)**.** This is a straightforward computation, such as the ones in Problems (5) and (6). To get the mean curvature, take $-\operatorname{tr} dN_p/2$.

Problem (8)**.** A closed curve encloses a simply connected region on the sphere, and we can apply Hopf's Umlaufsatz, which implies that the turn angle is 2π along a parallel, as well as along the curve in Part (b).

Problem (9)**.** See the top of p. 191.

Problem (10)**. This problem might also be too hard for an actual exam, but quite instructional, and simpler problems using these ideas will be on the actual exam.**

If a geodesic $\alpha(s)$ through p is planar, then it must be a line of curvature. To show that, let us note that the unit normal $N(\alpha(s))$ to the surface along a geodesic α is a unit normal for the geodesic, just like its canonical normal $N_\alpha = \alpha''/||\alpha''||$, and compute $dN_p(\alpha'(0))$. This way we will see that $\alpha'(0)$ is an eigenvector of dN_p. That is easy:

$$dN_p(\alpha'(0)) = d/dsN(\alpha(s)) = \pm d/dsN_\alpha(\alpha(s)) = \pm(-\kappa T + \tau B) = \pm\kappa T = \pm\kappa\alpha'(0).$$

Here we used the Frenet-Serret formulas and the fact that planar curves are torsion-free. Thus, $\alpha'(0)$ is an eigenvector and must be a principal direction. Since all the geodesics through p are planar, all the tangent directions at p are principal. Thus, the principal curvatures must be equal, and the point p must be umbilic.

Thus, all points p of the surface are umbilic and dN_p is diagonal with negative principal curvature $-k$ on the diagonal. Let us show that this implies that the surface is contained in either a plane or a sphere. First of all, from $dN_p \circ I_p = -II_p$, we get $e/E = f/F = g/G = k$. In a uv-parametrization of the surface, directions x_u and x_v at point p are also principal, and we have

$$dN_p(x_u) = kx_u \qquad \text{and} \qquad dN_p(x_v) = kx_v.$$

Recall that by definition of the differential map, $dN_p(x_u) = \partial/\partial uN(x(u,v))$ and $dN_p(x_v) = \partial/\partial vN(x(u,v))$. Since we are dealing with smooth maps, the mixed partials must be equal:

$$k_v x_u + kx_{uv} = k_u x_v + kx_{vu},$$

whence $k_v x_u = k_u x_v$. But x_u and x_v are linearly independent, forming a basis of the tangent plane to the surface at each point. Therefore, $k_u = k_v = 0$, which implies that k is constant.

Case $k = 0$. Then $dN_p = 0$ for all point p. We had a homework problem that this implies that the surface is part of a plane.

Case $k \neq 0$. Then take $C(u,v) := x(u,v) - (k)^{-1}N(x(u,v))$. We anticipate that this is the common center of the osculating circles of normal cross-sections at

$x(u,v)$ (It is useful to keep in mind what we are supposed to show at the end of the day!) and want to see that it stays put in space. Take the partials again:

$$\partial/\partial u C(u,v) = x_u - k^{-1}N_u = x_u - k^{-1}kx_u = 0$$

by the above, and similarly

$$\partial/\partial v C(u,v) = 0.$$

Hence, $C(u,v) = C$ is a constant point. Then $||x(u,v) - C|| = |1/k|$ and every point $x(u,v)$ lies at distance $|1/k|$ from C. Hence $x(u,v)$ lies on a sphere.

Problem (11)**.** This is done by finding the Christoffel symbols and solving the geodesic equations, which are simpler than those for a general conic surface. Look for solutions in the form $u = u(v)$ or $v = v(u)$.

Problem (12)**.** Compute the Jacobi matrix of the defining equations:

$$\begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 \\ 0 & 0 & 2x_3 & 2x_4 \end{pmatrix}$$

Because of the defining equations, both rows are nonzero, and are therefore linearly independent. Thus, this is a smooth submanifold of dimension $4 - 2 = 2$ in $\mathbb{R}^4$ by the corresponding theorem (which is basically the implicit function theorem).