Math 5615 Honors: Sequential compactness Cauchy sequences

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Sequential Compactness

Definition

Let X be a metric space. A subset $K \subset X$ is *sequentially compact* if every sequence in K has a subsequence that converges to a point in K.

Compare to the *Bolzano-Weirestrass property*: every infinite subset of K has a limit (cluster) point in K.

Theorem

$K \subset X$ TFAE:

1. *K is compact;*
2. *K is sequentially compact;*
3. *K has the B-W property.*

$$\begin{tikzcd} (1) \arrow[r, Rightarrow, "10/9"] \arrow[dr, Rightarrow, "10/2"'] & (2) \arrow[d, Rightarrow, "\text{next slide}"] \\ & (3) \arrow[ul, Rightarrow, bend left, "\text{Midterm}"] \end{tikzcd}$$

Proof. (1) $\Rightarrow$ (2): Previous theorem.
(1) $\Rightarrow$ (3): A theorem proven last Friday, 10/02/2020.
(3) $\Rightarrow$ (1): A problem on the Midterm.

Proof of Compactness Criterion, continued

The simplest thing to be done now: Show (2) $\Rightarrow$ (3).

Sppse $K \supset E$, ∞ subset. Hence $\exists$ a sequence in K with ∞ many elements in E, none of which repeated.

K seq. compact $\Rightarrow \exists$ subsequence converging to a point in K. This pt must be a limit pt of E.

(E.g., using the defn of a limit pt of E as having ∞ many pts of E in every open ball at the limit pt.) $\square$

Cauchy Sequences

Definition

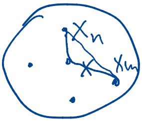
A sequence $\{x_k\}$ in a metric space X is a *Cauchy sequence* if for every $\varepsilon > 0$, there is a natural N such that if $m, n \geq N$, then $d(x_m, x_n) < \varepsilon$.

Theorem

A sequence that converges is necessarily a Cauchy sequence.

Proof. $\exists\, x \in X : \forall \varepsilon > 0\ \exists\, N > 0 : \forall n \geq N\ \ d(x_n, x) < \varepsilon/2$. Then $\forall\, m, n \geq N$ we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$



Complete Spaces

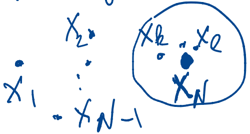
Definition

A metric space X is said to be *complete* if every Cauchy sequence in X has a limit in X.

Theorem

$\mathbb{R}^n$ is complete.

Proof. Step 1: Any Cauchy sequence $\{x_k\}$ in X is bounded. Let $\varepsilon = 1$. Then $\exists N : \forall k, l \geq N$ $d(x_k, x_l) < 1$. Take $M := \max\{d(x_1, x_j) \mid 1 \leq j \leq N-1\} \cup \{d(x_1, x_N) + 1\}$ (>0). Then $\forall x_k, k \geq 1$, $x_k \in B_M(x_1)$.



Step 2. Prove a version of B–W theorem: ~~every~~ bdd sequence $\{x_k\}$ ~~in $\mathbb{R}^n$~~ in $\mathbb{R}^n$ contains a convergent subsequence.

Follows from compactness of closed n-cell.

An n-cell is therefore sequentially compact

$\Rightarrow \{x_k\}$ has a subsequence $\{x_{n_k}\}$ converging to a point in the cell (and in $\mathbb{R}^n$).

Step 3. Let L = the limit of this subsequence.

Claim: Then the whole sequence has L as a limit.

Given $\varepsilon > 0$ take N_1: $|x_{n_k} - L| < \frac{\varepsilon}{2}$ if $k \geq N_1$

and take N_2: $|x_k - x_\ell| < \frac{\varepsilon}{2}$ $\forall\, k, \ell \geq N_2$. Then $\forall k \geq \max(N_1, N_2)$

$|x_k - L| \leq |x_k - x_{n_k}| + |x_{n_k} - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$ (note $n_k \geq k$ $\forall k$)

Closed Subsets of Complete Spaces

Theorem

If X is a complete metric space with respect to a metric d and Y is a nonempty closed subset of X, then Y is a complete metric space with the same metric d.

$\emptyset \neq Y \subset X$

Proof. Let $\{x_k\}$ be Cauchy seq. in Y, Then $\{x_k\}$ is also Cauchy in X. $\Rightarrow \exists \lim_{k\to\infty} x_k = x \in X$, Then $\forall \varepsilon > 0\ \exists N: \forall k \geq N\ d(x_k, x) < \varepsilon$.

$\Rightarrow$ every open ball about x contains a pt in Y,

$\Rightarrow x \in Y$ or is a cluster pt of Y. Since $Y \neq \emptyset$ and closed, $x \in Y$ $\square$

Compactness Implies Completeness

Theorem

Completeness is necessary for compactness.

Proof. X compact $\Rightarrow$ X complete.

A (Cauchy) sequence in a compact X has a subsequence convergent to $x \in X$.

$\Rightarrow$ The ~~whole~~ Cauchy sequence converges to x, just as in step 3 of the proof of completeness of $\mathbb{R}^n$. $\square$