Math 5615 Honors: Monotone sequences//Connectedness

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Monotone Sequences

Definition

A sequence $\{a_k\}$ of real numbers is *monotone increasing* if $a_{k+1} \geq a_k$ for all k, and *monotone decreasing* if $a_{k+1} \leq a_k$ for all k. A sequence is *monotone* if it is either monotone increasing or monotone decreasing.

Theorem

If a sequence in $\mathbb{R}$ is monotone and bounded, then it converges.

Proof. WLOG, assume $\{a_k\}$ is increasing. (Otherwise, $\{-a_k\}$ is.)

Claim: $\lim_{k\to\infty} a_k = L := \sup_k\{a_k\}$. (If decreasing then $\lim_{k\to\infty} a_k = L := \inf_k\{a_k\}$.)

Given $\varepsilon > 0$ $\exists N$: $L - \varepsilon < a_N \leq L$. Then $\forall n \geq N$

$L - \varepsilon < a_N \leq a_n \leq L \Rightarrow \underset{d(a_n, L)}{|a_n - L|} < \varepsilon$. $\square$

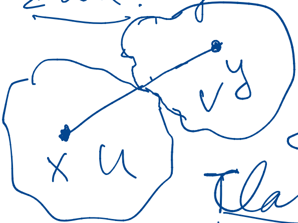
A Convex Set in $\mathbb{R}^n$ Is Connected

(HW problem) $C \subset \mathbb{R}^n$ convex

($C \subseteq U \cup V$)

<u>By contradiction</u>: Suppose $C = (U \cup V) \cap C$, U, V open in $\mathbb{R}^n$, $(U \cap C) \cap (V \cap C) = \emptyset$, $U \cap C \neq \emptyset$, $V \cap C \neq \emptyset$, has a separation.

<u>Idea</u>: get a separation of $[0,1]$.



Take $x \in U \cap C$, $y \in V \cap C$

Then $[x,y] = \{\lambda x + (1-\lambda) y \mid \lambda \in [0,1]\} \subset C$, b/c C convex.

<u>Claim</u>: $U' = \{\lambda \in [0,1] \mid \lambda x + (1-\lambda) y \in U\}$ and $V' = \{\lambda \in [0,1] \mid \lambda x + (1-\lambda) y \in V\}$ separate $[0,1]$, i.e., $U' \neq \emptyset \neq V'$, $U' \cap V' = \emptyset$, $U' \cup V' = [0,1]$, and U', V' open in $[0,1]$.

A Convex Set in $\mathbb{R}^n$ Is Connected

U' is open in $[0,1]$:

(If we knew conts. fnctns:

$f: [0,1] \to C$

$f(\lambda) := \lambda x + (1-\lambda) y$

conts, being linear

$U' = f^{-1}(U)$ by construction of U'

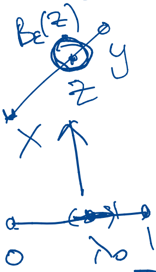
and therefore U' is open as pre-image of open under conts map.)

Instead, show U' is open directly.

A Convex Set in $\mathbb{R}^n$ Is Connected

Every pt $\lambda_0 \in U'$ contains an open ball lying in U' and centered at λ_0,
which will be in U, b/c $\lambda_0 \in U'$.

Take $\vec{z} = \lambda_0 x + (1-\lambda_0) y$, Know $\exists\ \varepsilon > 0$: $B_\varepsilon(z) \subset U$



$$B_\varepsilon(z) \cap [x,y] = \{\lambda x + (1-\lambda) y \mid \underbrace{|\lambda x + (1-\lambda) y - (\underbrace{\lambda_0 x + (1-\lambda_0) y}_{z})|}_{=\ |(\lambda-\lambda_0)x - (\lambda-\lambda_0)y| \ =\ |\lambda-\lambda_0|\cdot|x-y|} < \varepsilon\}$$

Take $\delta = \frac{\varepsilon}{|x-y|+1}$ (it's >0). Then

if $|\lambda-\lambda_0| < \delta$ then $|\lambda-\lambda_0|\cdot|x-y| \leq |\lambda-\lambda_0|\cdot(|x-y|+1) < \delta\cdot(|x-y|+1) = \varepsilon$, i.e., $\lambda x + (1-\lambda) y \in B_\varepsilon(z) \subset U$ and $\lambda \in U'$.

A Convex Set in $\mathbb{R}^n$ Is Connected

Could have worked with opens in $\mathbb{R}$ instead of $[0,1]$:

$$U' = \{ \lambda \in \mathbb{R} \mid \lambda x + (1-\lambda) y \in U \}$$

etc. $\quad U' \subset \mathbb{R}$ open

$U' \cap [0,1] \neq \emptyset$, $V' \cap [0,1] \neq \emptyset$,

$$(U' \cap [0,1]) \cup (V' \cap [0,1]) = [0,1]$$

$$(U' \cap [0,1]) \cap (V' \cap [0,1]) = \emptyset$$