Math 5615 Honors: Some Special Sequences

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Proposition

If a is a real or complex number and $|a| < 1$, then

$$\lim_{n\to\infty} a^n = 0.$$

Proof.

Step 1: The $a = 0$ case.

$$\lim_{n\to\infty} 0^n = \lim_{n\to\infty} 0 = 0$$

Step 2: If $0 < |a| < 1$, then $\{|a^n|\}$ is monotone decreasing.
$\exists L := \lim_{n\to\infty} |a^n| \geq 0$.

$$|a^{n+1}| = |a| \cdot |a^n| < |a^n|, \text{ b/c}$$

$$|a| < 1, \quad |a^n| > 0$$

Continuing with $\lim_{n\to\infty} a^n$

Step 3: The limit $L = 0$. Idea: Suppose $L > 0$. Then

$$|a^n| = \frac{|a^{n+1}|}{|a|} \geq \frac{L}{|a|} \qquad \forall n \geq 1,$$

and $L/|a| > L$ is also a lower bound. Contradicts

$L = \inf_n \{|a|^n\}$, greatest lower bound.

Step 4: $\lim_{n\to\infty}|a^n| = 0 \Leftrightarrow \lim_{n\to\infty} a^n = 0$. We see that

$||a^n| - 0| = |a^n| = |a^n - 0| < \varepsilon$

simultaneously.

Cf.: The textbook uses Bernoulli's inequality $(1+h)^n \geq 1 + nh$ for $h > 0$, which we will use for the next special limit.

Proposition

If $a > 0$, then $\lim_{n\to\infty} a^{1/n} = 1$.

Proof.

Step 1: The $a = 1$ case.

$\lim_{n\to\infty} (1)^{\frac{1}{n}} = \lim 1 = 1$

Step 2: If $a > 1$, then $a^{1/n} > 1$. Set $b_n = a^{1/n} - 1$. Then $\lim_{n\to\infty} b_n = 0$, which is equivalent to $\lim_{n\to\infty} a^{1/n} = 1$.

$a > b \geq 0 \Rightarrow \sqrt[n]{a} > \sqrt[n]{b} \geq 0$

(otherwise $0 \leq x \leq y \Rightarrow x^n \leq y^n$ $\quad x \cdot x \leq x \cdot y \leq y \cdot y$ etc.)

$a = (1 + b_n)^n \geq 1 + n b_n$ (Bernoulli's inequality see HW)

$\Rightarrow 0 < b_n \leq \frac{a-1}{n} \xrightarrow[\text{HW}]{\text{squeeze thm}} \lim_{n\to\infty} b_n = 0$

Step 3: If $0 < a < 1$, then $1/a > 1$ and by Step 2

$$\lim_{n\to\infty} (1/a)^{1/n} = 1.$$

Use the quotient rule:

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} \frac{1^{1/n}}{(1/a)^{1/n}} = \frac{1}{1} = 1.$$

$\square$

The Euler Number e

Theorem

The sequence $\{(1+1/n)^n\}$ is increasing and convergent. The limit is called the Euler number e:

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

This is to answer Nhung's question.

Proof.

$\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \xRightarrow{(1+\frac{1}{x})^x \text{ conts fnctn}} \lim_{n\to\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = e, \text{ if } x_n \to \infty\right]$

Lemma

Let a and b be real numbers such that $0 \le a < b$. Then

$$\frac{b^{n+1}-a^{n+1}}{b-a} < (n+1)b^n.$$

$\left(\underline{\text{Cf.}}\ (x^{n+1})' = (n+1)x^n\right)$

Proof of Lemma:

$$\frac{b^{n+1}-a^{n+1}}{b-a} = \frac{(b-a)(b^n + b^{n-1}a + b^{n-2}a^2 + \dots + a^n)}{b-a} < b^n + b^n + b^n + \dots + b^n = (n+1)b^n.$$

Lemma proven.

Proof of the Euler Number Theorem

Step 1: Sequence is increasing. Set $a = 1 + 1/(n+1)$ and $b = 1 + 1/n$ in Lemma.

$$\frac{\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1}}{\frac{1}{n} - \frac{1}{n+1}} < (n+1)\left(1+\frac{1}{n}\right)^{n}$$

$$\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n+1}\right)^{n+1} < \frac{1}{n}\left(1+\frac{1}{n}\right)^{n}$$

$$\left(1+\frac{1}{n}\right)^{n}\underbrace{\cancel{\left(1+\frac{1}{n}-\frac{1}{n}\right)}}_{1} < \left(1+\frac{1}{n+1}\right)^{n+1}$$

Continuing with Proof of the Limit for e

Step 2: Sequence is bounded. Set $a = 1$ and $b = 1 + 1/2n$ in Lemma.

$$\frac{\left(1+\frac{1}{2n}\right)^{n+1} - 1}{\left(\frac{1}{2n}\right)} < (n+1)\left(1+\frac{1}{2n}\right)^{n}$$

$$\left(1+\frac{1}{2n}\right)^{n+1} - 1 < \frac{n+1}{2n}\left(1+\frac{1}{2n}\right)^{n}$$

$$\left(1+\frac{1}{2n}\right)^{n}\left(1+\frac{1}{2n} - \frac{n+1}{2n}\right) < 1$$

$$\left(1+\frac{1}{2n}\right)^{n}\cdot\frac{1}{2} < 1, \quad \left(1+\frac{1}{2n}\right)^{n} < 2 \overset{\text{square}}{\Rightarrow} \left(1+\frac{1}{2n}\right)^{2n} < 4$$

$$\left(1+\frac{1}{n}\right)^{n} \underset{\text{increasing}}{<} \left(1+\frac{1}{2n}\right)^{2n} < 4 \quad \forall n \geq 1 \qquad \square$$