Math 5615H: Honors: Introduction to Analysis
More on Compact Sets
Nested Intervals and Heine-Borel
The Cantor Set

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The Bolzano-Weierstrass Property

Definition

Let X be a metric space. A subset $A \subset X$ has the *Bolzano-Weierstrass property* if every infinite subset of A has a limit point (cluster point) that belongs to A.

Theorem

Let A be a subset of a metric space (X, d). Then A is compact iff A has the Bolzano-Weierstrass property.

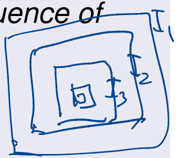
Proof.

$\Rightarrow$: Proved last time.
$\Leftarrow$: Part of the upcoming take-home Midterm Exam 1. $\square$

Nested Intervals

Theorem (Nested interval property)

If $I_k = [a_1^{(k)}, b_1^{(k)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}]$ is a nested sequence of n-cells in $\mathbb{R}^n$, that is, if



$$I_1 \supset I_2 \supset \cdots \supset I_k \supset I_{k+1} \supset \ldots,$$

then the intersection $\bigcap_{k=1}^{\infty} I_k$ is nonempty. If $|b_j^{(k)} - a_j^{(k)}| < \frac{1}{k}$ for each $j = 1, 2, \ldots, n$, then the intersection consists of a single point.

Proof. 1. The $n = 1$ case:

$a_k \le a_\ell \le b_\ell \le b_k$ for $k, \ell \in \mathbb{N}$, $\ell > k$

$A = \{a_k \mid k \in \mathbb{N}\}$ bdd above (by b_1) $\Rightarrow x = \sup A$

$x \le b_k \ \forall k$, $x \ge a_k \ \forall k$ $\Rightarrow x \in \bigcap_{k=1}^{\infty} [a_k, b_k]$

Uniqueness:

$B = \{b_k \mid k \in \mathbb{N}\}$, $y = \inf B$, $a_k \le x \le y \le b_k$ $\forall k$

$y \in \bigcap_{k=1}^{\infty} [a_k, b_k]$

check that y is an upper bd for A

If $\forall k$ $|b_k - a_k| < \frac{1}{k}$, then $|y - x| < \frac{1}{k}$. Lemma $\Rightarrow y = x$.

$[x, y] = \bigcap_{k=1}^{\infty} [a_k, b_k]$

2. The $n \geq 2$ case:

$\forall j = 1, \dots, n$ get a nested sequence of intervals of $\mathbb{R}$:

$\dots \supset [a_j^{(k)}, b_j^{(k)}] \supset [a_j^{(k+1)}, b_j^{(k+1)}] \supset \dots$

$\exists c_j \in \bigcap_{k \geq 1} [a_j^{(k)}, b_j^{(k)}]$. Then $(c_1, \dots, c_n) \in \bigcap_{k \geq 1} I_k$

$I_k = \{(x_1, \dots, x_n) \mid a_1^{(k)} \le x_1 \le b_1^{(k)}, \dots, a_n^{(k)} \le x_n \le b_n^{(k)}\}$

If there is a bd by $\frac{1}{k}$ on ~~the~~ the "size" of I_k, then each c_i, $i = 1, \dots, n$, will be unique.

The Bolzano-Weierstrass Theorem

Theorem

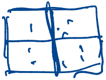
A bounded infinite set S in $\mathbb{R}^n$ has at least one cluster point (which need not be an element of S).

Proof. Step 1: Construct nested n-cells.

S bdd, $\infty \subset \mathbb{R}^n \Rightarrow \exists$ cluster pt of S in $\mathbb{R}^n$

S bdd $\Rightarrow \exists\, I_0 \supset S$ n-cell

Divide I_0 into 2^n smaller cells by bisecting each side of I_0:



I_0

$\exists$ subcell containing ∞ many pts of S. Call it I_1.

Repeat: subdivide I_1 into 2^n subcells, choose I_2 to contain ∞ many pts of S. ... $\Rightarrow$ Nested sequence of n-cells: $I_0 \supset I_1 \supset I_2 \supset \dots$, $M :=$ the longest side of I_0

The Proof of the Bolzano-Weierstrass Theorem, Continued

Then the sides of I_k are bdd $(<)$ by $\frac{M+1}{2^k}$.
Like in the nested intervals then,
$\exists !$ pt x_0 in $\bigcap_{k=1}^{\infty} I_k = \{x_0\}$.

The Proof of the Bolzano-Weierstrass Theorem, Completed

Step 2: **Claim**: The common point is a cluster point of S.

Claim x_0 is a cluster pt of S.

$B_\varepsilon(x_0)$ Find $I_k \subset B_\varepsilon(x_0)$

(e.g., take k: $\frac{M}{2^k} < \frac{\varepsilon}{\sqrt{n}}$) Then $\exists \infty$ many pts of S in $I_k \Rightarrow$ in $B_\varepsilon(x_0)$. $\square$

The Heine-Borel Theorem

Theorem

A subset K of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. $\Rightarrow$: Proved for any metric space instead of $\mathbb{R}^n$.
$\Leftarrow$: If K is bounded, then $K \subset I$ for some n-cell I. Since K is closed, it is enough to show I is compact. $I_0 := I$

Lemma

Every n-cell I is compact. $I_0 := I$

Proof of Lemma. By contradiction: suppose there is an open cover $\{O_\alpha\}$ which does not admit a finite subcover. Bisect ~~and~~ I_0 and construct $I_1 \supset I_2 \supset \dots$ of n-cells each of which does not have a finite subcover. Should continue indefinitely. But then the unique common point x is in some O_{α_0}. Since it is open, there is some $B_\delta(x) \subset O_{\alpha_0}$. But I_k for large enough k will be in $B_\delta(x)$ and I_k is covered by just one O_{α_0}. Lemma is proven. $\square$